MODELING OF DYNAMIC SYSTEMS
1^{st} homework

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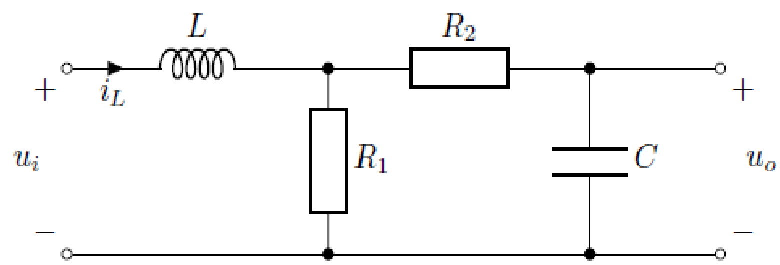
1 Electrical circuit

Default system parameters:

- $R_1 = 50\ \Omega$,
- $R_2 = 60\ \Omega$,
- L = 0.04 H,
- C = 1.5 µF.



1.1 Differential equation

To obtain the final form of the desired differential equation, several equations need to be determined beforehand. From the I Kirchhoff's law:

$$i_{R_2}(t) = i_C(t) = i(t),$$

$$i_L(t) = i_{R_1}(t) + i_{R_2}(t) = i_{R_1}(t) + i(t).$$

From the II Kirchhoff's law:

$$u_i(t) = u_L(t) + u_{R_1}(t) = L\frac{di_L(t)}{dt} + i_{R_1}(t)R_1,$$

$$u_{R_2}(t) + u_C(t) = u_{R_1}(t) \Leftrightarrow i_{R_2}(t)R_2 + \int \frac{i_C(t)}{C}dt + u_C(0) = i_{R_1}(t)R_1.$$

We can also observe, that:

$$u_o(t) = u_C(t) = \int \frac{i_C(t)}{C}dt + u_C(0).$$

Having all those equations, we can make some substitutions and transformations, final equation can be obtained:

$$i_C(t) = i(t)$$

$$u_o(t) = \int \frac{i(t)}{C} dt + u_C(0) \rightarrow C\frac{du_o(t)}{dt} = i(t) \rightarrow C\frac{d^2u_o(t)}{dt^2} = \frac{di(t)}{dt}$$

$$i_{R_1}(t)R_1 = i(t)R_2 + \int \frac{i_C(t)}{C} dt + u_C(0) \rightarrow i_{R_1}(t) = \frac{1}{R_1}\left(C\frac{du_o(t)}{dt}R_2 + u_o(t)\right)$$

$$u_i(t) = L\frac{d(i(t) + i_{R_1}(t))}{dt} + i_{R_1}(t)R_1 \rightarrow u_i(t) = L\left(C\frac{d^2u_o(t)}{dt^2} + \frac{1}{R_1}\left(C\frac{d^2u_o(t)}{dt^2}R_2 + \frac{du_o(t)}{dt}\right)\right) + R_2C\frac{du_o(t)}{dt} + u_o(t)$$

After some simplifications:

$$u_i(t) = \frac{d^2u_o(t)}{dt^2}LC\left(1 + \frac{R_2}{R_1}\right) + \frac{du_o(t)}{dt}\left(\frac{L}{R_1} + R_2C\right) + u_o(t)$$

1.2 Stationary values

From previous equations, it is already known that $u_o(t) = u_C(t)$. Having that in mind, all output voltages (and their derivates) can be substituted with corresponding $U_C(t)$ values:

$$u_i(t) = \frac{d^2u_C(t)}{dt^2}LC\left(1 + \frac{R_2}{R_1}\right) + \frac{du_C(t)}{dt}\left(\frac{L}{R_1} + R_2C\right) + u_C(t).$$

To obtain a stationary equation, replacing all derivatives with 0 needs to be performed. Therefore:

$$u_{i0} = u_{C0}.$$

To find the i_{L0} dependence of u_{i0}, $i_L(t) = i(t) + i_{R_1}(t)$ equation can be used. As:

$$i(t) = C\frac{du_o(t)}{dt},$$

$$i_{R_1}(t) = \frac{1}{R_1}\left(C\frac{du_o(t)}{dt}R_2 + u_o(t)\right),$$

$$u_o(t) = u_i(t) - \frac{d^2u_o(t)}{dt^2}LC\left(1 + \frac{R_2}{R_1}\right) - \frac{du_o(t)}{dt}\left(\frac{L}{R_1} + R_2C\right),$$

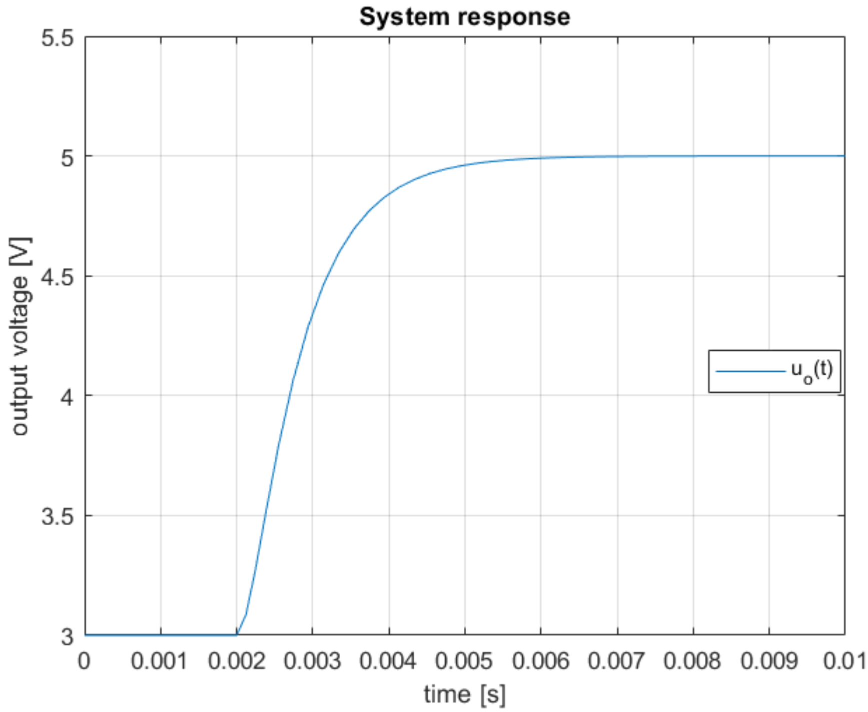
after combining those equations and substituting derivtives with 0, the final dependence is:

$$i_{L0} = \frac{1}{R_1}u_{i0}.$$

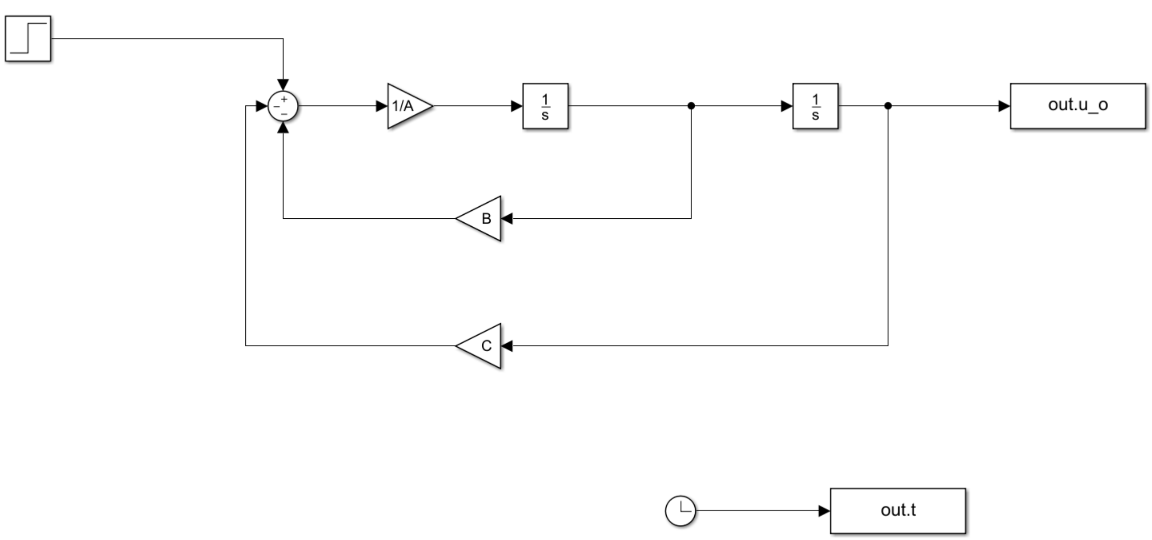
Static characteristics of the system are not sufficient to infer the linearity of the system - as the derivatives change into 0 at stationary equation, everything which is multiplied by them "disappears". Disappeared elements can be nonlinear (exponentials, square roots, derivatives etc.), while their nonlinearity cannot be observed in the static characteristics.

1.3 System's simulation

Input signal: $u_i(t) = (3 + 2S(t - 0.002))$.
Stationary values corresponding to $u_{i0} = 3V$.



Plotted time response



Simulink model

```
close all;
clear;

L = 0.04;
C = 1.5 * 10^(-6);
R_1 = 50;
R_2 = 60;
A = L*C*(1 + R_2/R_1);
B = L/R_1 + R_2*C;
C = 1;

u_i0 = 3;
u_o1 = 0;
u_o0 = u_i0;

s_time = 0.002;
delta = 2;
i_value = u_i0;
sim_time = 0.01;

sim('model1');
figure;
plot(ans.t, ans.u_o);
grid on;
axis([0 sim_time i_value i_value+delta+0.5]);
xlabel('time [s]');
ylabel('output voltage [V]');
legend('u_o(t)');
title('System response');
```

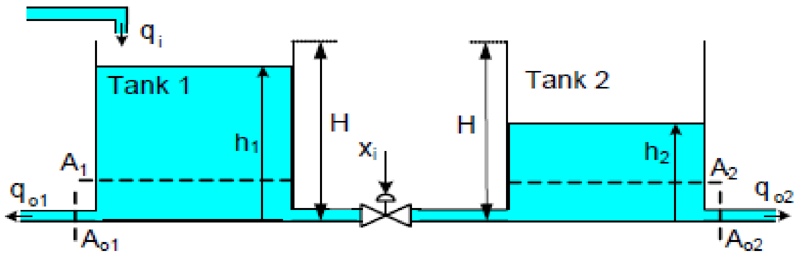
Matlab code

2 Fluid storage system

2.1 Differential equations

Default system parameters:

- $q_i = 30\frac{kg}{s}$,
- $A_1 = 2m^2$,
- $A_2 = 3m^2$,
- $A_v = 0.001m^2$,
- $A_{o1} = 0.002m^2$,
- $A_{o2} = 0.001m^2$,
- $\rho = 1000\frac{kg}{m^3}$,
- H = 10 m,
- $g = 9.81\frac{m}{s^2}$.



For the first tank, it can be seen that the amount of fluid at some specific time is described by the equation:

$$A_1\rho\dot{h_1}(t) = q_i(t) - A_{o1}\rho\sqrt{2gh_1(t)} - sgn(h_1(t) - h_2(t)) \cdot A_v\rho x_i(t)\sqrt{2g|h_1(t) - h_2(t)|}.$$

Signum function is put to distinguish, whether the fluid flow is directed from container nr 1 to nr 2 or reversely. The behaviour of the second tank can be described as:

$$A_2\rho\dot{h_2}(t) = sgn(h_1(t) - h_2(t)) \cdot A_v\rho x_i(t)\sqrt{2g|h_1(t) - h_2(t)|} - A_{o2}\rho\sqrt{2gh_2(t)}.$$

2.2 Stationary values

Let's assume, that:

$$a_1 = A_{o1}\rho\sqrt{2g},$$

$$a_2 = A_{o2}\rho\sqrt{2g},$$

$$a_v = A_v\rho\sqrt{2g}.$$

Then the equations have the following form:

$$A_1\rho\dot{h_1}(t) = q_i(t) - a_1\sqrt{h_1(t)} - sgn(h_1(t) - h_2(t)) \cdot a_v x_i(t)\sqrt{|h_1(t) - h_2(t)|},$$

$$A_2\rho\dot{h_2}(t) = sgn(h_1(t) - h_2(t)) \cdot a_v\rho x_i(t)\sqrt{|h_1(t) - h_2(t)|} - a_2\sqrt{h_2(t)}.$$

After replacing every derivative with 0, values of h_{10} and h_{20} need to be found. After some transformations, with the assumption that $h_{10} > h_{20}$:

$$h_{20} = \frac{a_v^2 x_{i0}^2}{a_2^2 + a_v^2 x_{i0}^2} h_{10} = bh_{10},$$

$$h_{10} = \frac{q_i^2}{a_1^2 + 2a_1a_vx_{i0}\sqrt{1-b} + a_v^2x_{i0}^2(1-b)}.$$

In the opposite case:

$$h_{20} = \frac{a_v^2x_{i0}^2}{a_v^2x_{i0}^2 - a_2^2}h_{10} = bh_{10},$$

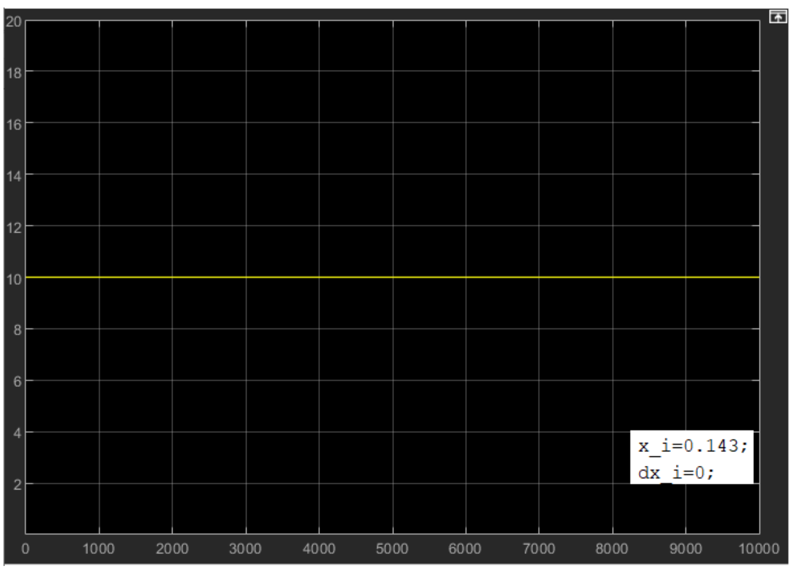
$$h_{10} = \frac{q_i^2}{a_1^2 - 2a_1a_vx_{i0}\sqrt{b-1} + a_v^2x_{i0}^2(b-1)}.$$

The truth is that in stationary state h_{10} will always have to be greater than h_{20} (the first case is relevant). It is because the fluid from the 2nd container will flow away through outlet pipe, so to remain stationary value of h_2, the fluid must flow into 2nd container through the valve. To have such situation, $h_{10} > h_{20}$.
To determine, for which stationary value x_{i0} the fluid overflow occurs, the following inequality has to be solved:

$$h_{10} > 10,$$

$$b\frac{q_i^2}{a_1^2 + 2a_1a_vx_{i0}\sqrt{1-b} + a_v^2x_{i0}^2(1-b)} > 10,$$

where $b = \frac{a_v^2x_{i0}^2}{a_2^2+a_v^2x_{i0}^2}$. Via trials and errors method (by changing x_{i0} values already in the simulation), the approximate value for which the fluid overflows has been found. A scope block in the Simulink was used to plot the h_{10} value.

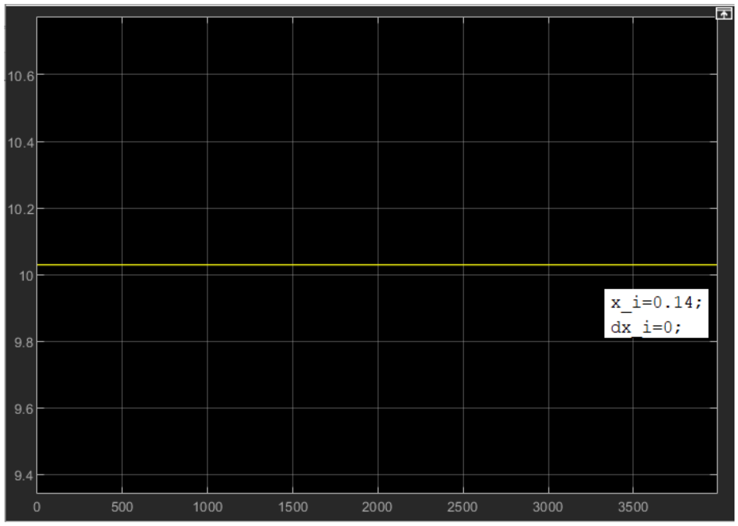


Scoped value of h_{20} for $x_{i0} = 0.143$

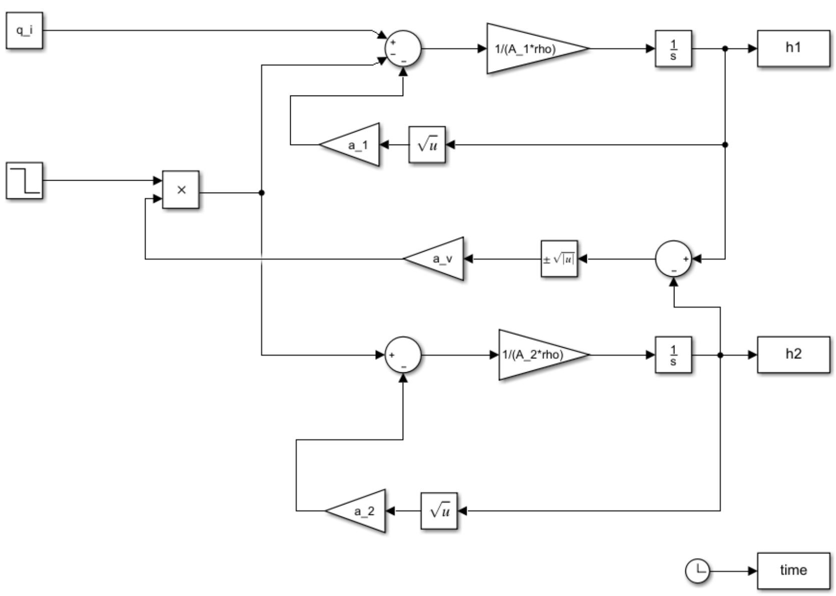
The input value x_{i0} is around 0.143. This way of finding the result is not a clean solution, it could have been calculated manually, more accurately, but it is sufficient enough in this case.

2.3 Simulink model

As mentioned above, it was tested for which value an overflow occurs. To picture that, $x_{i0} = 0.14$ has been set and run in the simulation.



Scoped value of h_{20} for $x_{i0} = 0.14$



Simulink model

```
clear all;
close all;

g = 9.81;
q_i = 30;
A_1 = 2;
A_2 = 3;
A_v = 0.001;
A_o1 = 0.002;
A_o2 = 0.001;
rho = 1000;
H = 10;

a_1 = A_o1 * rho * sqrt(2*g);
a_2 = A_o2 * rho * sqrt(2*g);
a_v = A_v * rho * sqrt(2*g);

t_sim = 10000;
X_0 = 0.5;
x_i_tab = [X_0 X_0+0.2 X_0 X_0+0.2];
dx_i_tab = [0.15 0.15 -0.15 -0.15];
step_time = 1000;

descr = '';

figure; hold on;
for i=1:length(x_i_tab)
    x_i=x_i_tab(i);
    dx_i=dx_i_tab(i);
    b = (a_v^2*x_i^2)/(a_2^2 + a_v^2 * x_i^2);
    h_10 = q_i^2/(a_1^2 + 2 * a_1 * a_v * x_i * sqrt(1-b) + a_v^2 * x_i^2 * (1-b));
    h_20 = b * h_10;
    sim('model2');
    plot(time, abs(h2-h_20));
    descr=[descr, sprintf('delta h_2 for %s %s S(t - %s),', string(x_i), string(dx_i), string(step_time) )];
end

xlabel('time [s]');
ylabel('height [m]');
title('Height of the fluid in 2nd tank');
grid on;


legend(strsplit(descr, ','));
```

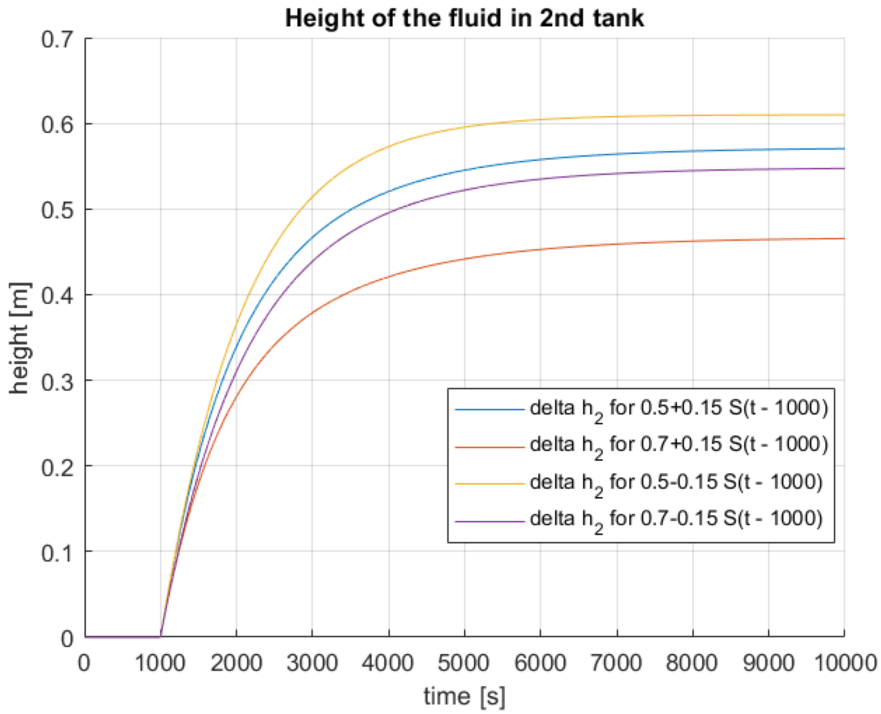
Matlab code

2.4 Simulations

In the simulation model for an arbitrarily selected valve opening $x_{i0} = 0.5$, the following cases have been simulated:

- $x_i(t) = x_{i0} \pm 0.15S(t$ - $1000)$,
- $x_i(t) = (x_{i0} + 0.2) \pm 0.15S(t$ - $1000)$.

The responses $|h_2(t)| = |h_2(t) - h_{20}|$ have been plotted on one graph.



Time responses for the height of the fluid in the 2nd tank

For a lower initial value, the height of the 2nd container takes longer to stabilize, but stabilizes on a higher absolute value than in the cases with higher initial values.