LINEARIZATION OF NONLINEAR DYNAMIC SYSTEMS
2^{nd} homework

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1 Second-order differential equation

A nonlinear second-order differential equation of our interest is described as:

$$\ddot{y}(t)y^2(t) + 3\dot{y}(t) + y(t)e^{2u(t)} = 1. \tag{1}$$

1.1 Linearization

To linearize the given equation around the operating point defined by $y_0 = 1$, following operations need to be done:

- establishing a function
$$f(\ddot{y}, \dot{y}, y, u) = \ddot{y}y^2 + 3\dot{y} + ye^{2u} - 1, \tag{2}$$
- determining all steady state values
$$y_0 = 1, \dot{y_0} = 0, \ddot{y_0} = 0, u_0 = 0, \tag{3}$$
- calculating partial derivatives
$$\left.\frac{\delta f}{\delta y}\right|_0 = 2\ddot{y_0}y_0 + e^{2u_0} = 1, \tag{4}$$
$$\left.\frac{\delta f}{\delta u}\right|_0 = 2y_0e^{2u_0} = 2, \tag{5}$$
$$\left.\frac{\delta f}{\delta \ddot{y}}\right|_0 = y_0 = 1, \tag{6}$$
$$\left.\frac{\delta f}{\delta \dot{y}}\right|_0 = 3, \tag{7}$$
- using Taylor series to determine linear equation
$$\Delta\ddot{y} + 3\Delta\dot{y} + \Delta y + 2\Delta u = 0. \tag{8}$$

1.2 Transfer function

The next step is to determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$, where $Y(s) = \mathcal{L}(\Delta y(t))$ and $U(s) = \mathcal{L}(\Delta u(t))$. If Laplace tranform is applied on equation [8], we obtain:

$$s^2 Y(s) + 3sY(s) + Y(s) = -2U(s). \tag{9}$$

After some tranformations:

$$G(s) = \frac{-2}{s^2 + 3s + 1}. \tag{10}$$

1.3 Simulations

Linearized and nonlinear models have been created using Simulink.

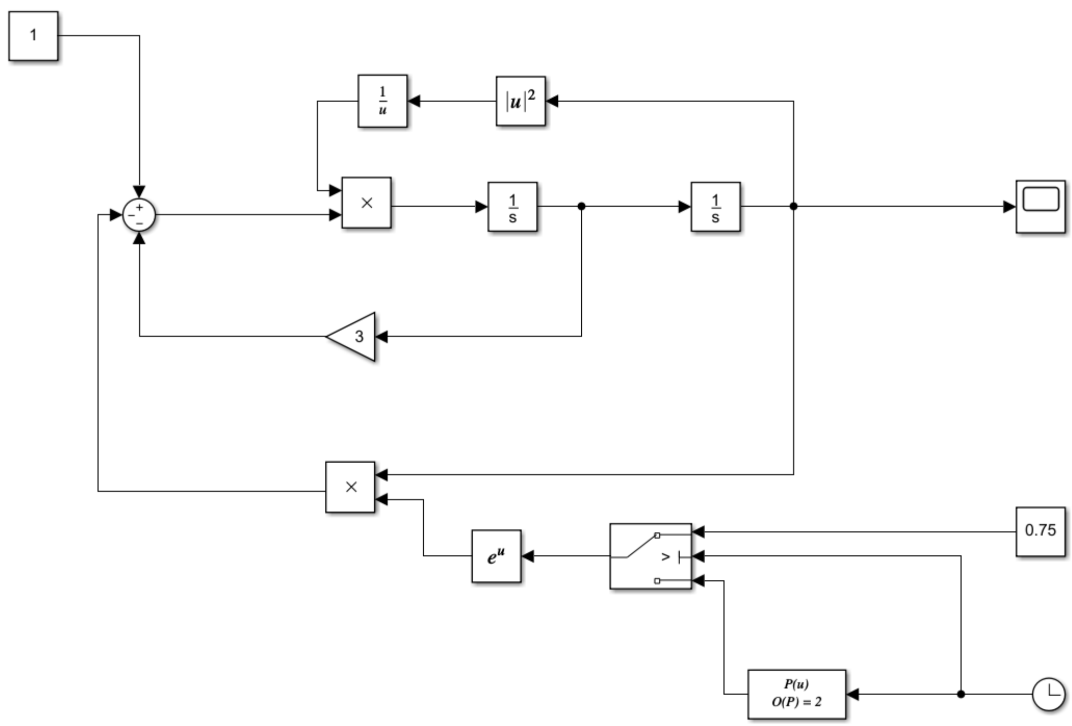


Figure 1: Nonlinear model.

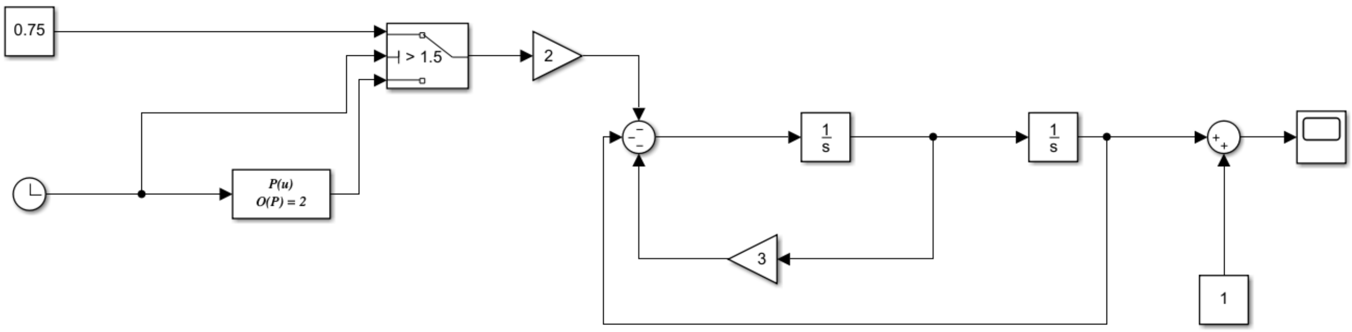


Figure 2: Linearized model.

The input of the system is $u_0 + \Delta(t)u$, where the function $\Delta u(t)$ is shown in the figure [3].

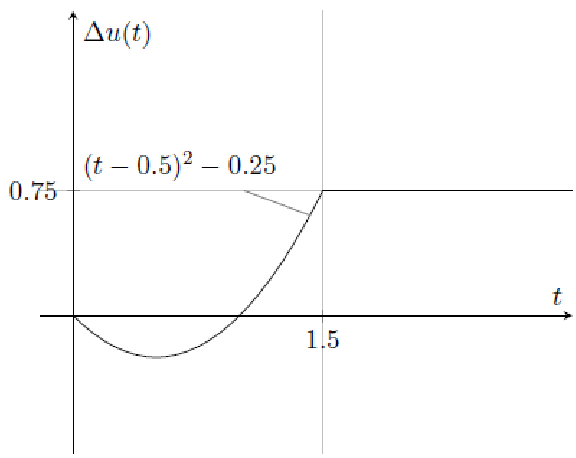


Figure 3: Function $\Delta u(t)$.

The response for both nonlinear and linearized systems are presented respectively in figures [4] and [5].

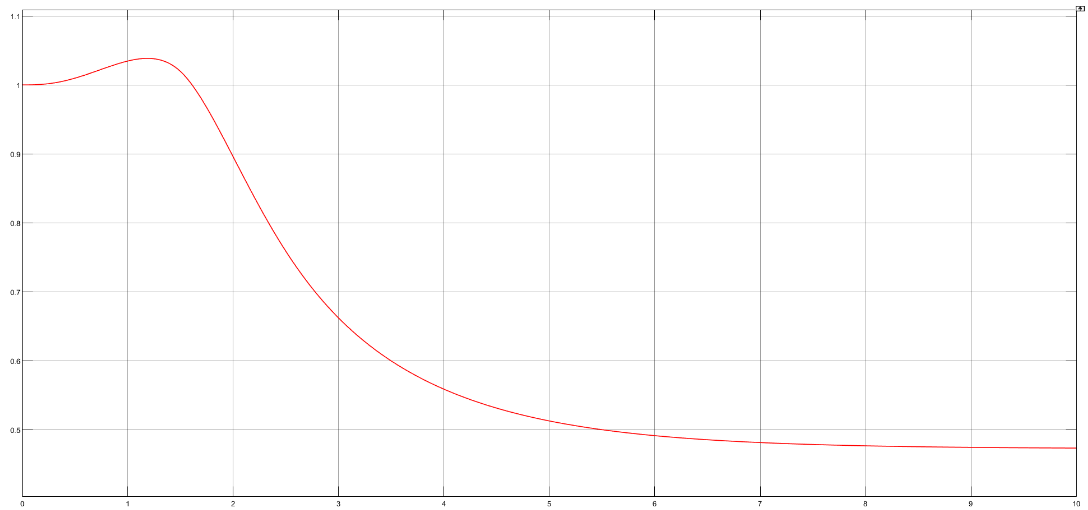


Figure 4: Nonlinear system's response.

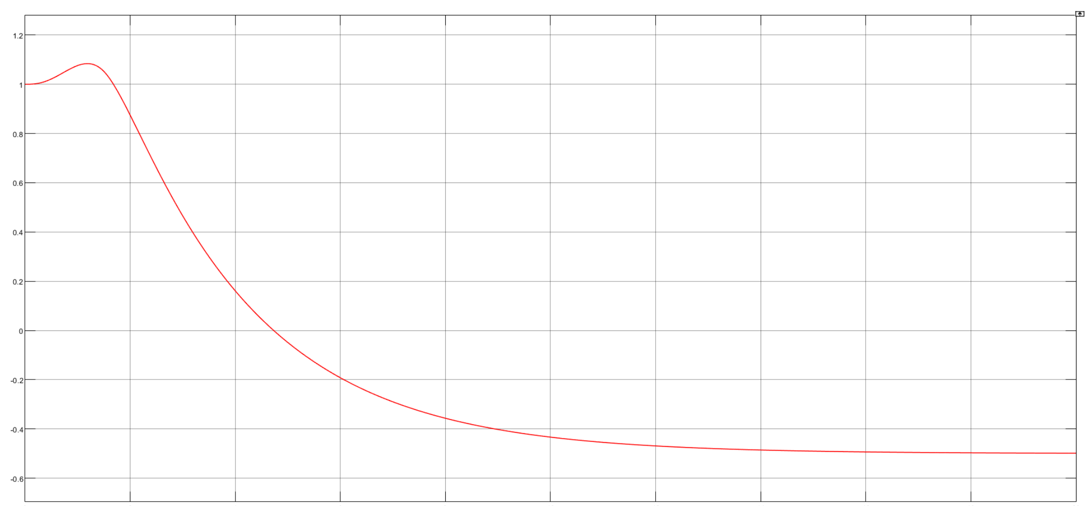


Figure 5: Linearized system's response.

An error in linearized model in the steady state occurs as the systems differ - if we look at steady state equations of both systems:

$$\Delta y_0 = -2\Delta u_0, \tag{11}$$

$$y_0 = \frac{1}{e^{2u_0}}, \tag{12}$$

then we see that they are not equivalent.

1.4 Further analysis

By replacing all derivatives with 0, we obatin the stationary value of the linearized system as shown in the equation [11]. The slope of that output at time instant $t = 0^+$ s is in fact the solution of the following formulas:

$$Y(s) = G(s) \cdot U(s) = \frac{-2}{s^2 + 3s + 1} \cdot \mathcal{L}(t^2 - t) = \frac{-2(s + 2)}{(s^2 + 3s + 1)s^3} \tag{13}$$

$$\Delta\dot{y}(0^+) = \lim_{t \to 0^+} \Delta\dot{y}(t) = \lim_{s \to \infty} s^2 Y(s) = \lim_{s \to \infty} s^2 \frac{-2(s + 2)}{(s^2 + 3s + 1)s^3} = 0 \tag{14}$$

By adding an additional scope block in Simulink, the following plot for $\Delta\dot{y}$ is obtained:

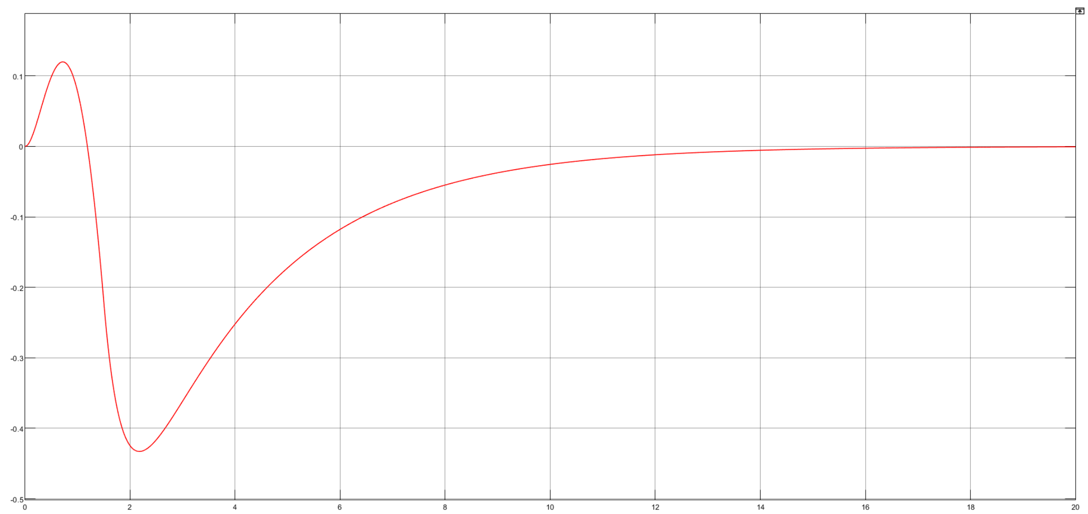


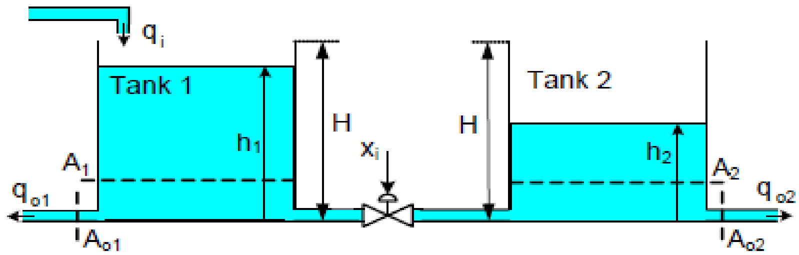
Figure 6: Linearized system's output's slope.

2 Fluid storage system

2.1 Differential equations

Default system parameters:

- $q_i = 30\frac{kg}{s}$,
- $A_1 = 2m^2$,
- $A_2 = 3m^2$,
- $A_v = 0.001m^2$,
- $A_{o1} = 0.002m^2$,
- $A_{o2} = 0.001m^2$,
- $\rho = 1000\frac{kg}{m^3}$,
- H = 10 m,
- $g = 9.81\frac{m}{s^2}$.



Let's assume, that:

$$a_1 = A_{o1}\rho\sqrt{2g}, \tag{15}$$

$$a_2 = A_{o2}\rho\sqrt{2g}, \tag{16}$$

$$a_v = A_v\rho\sqrt{2g}. \tag{17}$$

Then the equations describing the system have the following form:

$$A_1\rho\dot{h_1}(t) = q_i(t) - a_1\sqrt{h_1(t)} - sgn(h_1(t) - h_2(t)) \cdot a_v x_i(t)\sqrt{|h_1(t) - h_2(t)|}, \tag{18}$$

$$A_2\rho\dot{h_2}(t) = sgn(h_1(t) - h_2(t)) \cdot a_v \rho x_i(t)\sqrt{|h_1(t) - h_2(t)|} - a_2\sqrt{h_2(t)}. \tag{19}$$

In the stationary state $h_{10} > h_{20}$:

$$h_{20} = \frac{a_v^2 x_{i0}^2}{a_2^2 + a_v^2 x_{i0}^2} h_{10} = b h_{10}, \tag{20}$$

$$h_{10} = \frac{q_i^2}{a_1^2 + 2a_1 a_v x_{i0}\sqrt{1-b} + a_v^2 x_{i0}^2(1-b)}. \tag{21}$$

2.2 Linearization

To linearize the given equations around the operating point defined by $x_{i0} = 0.2$, following operations need to be done:

- establishing functions

$$A_1\rho\dot{h_1}(t) = q_i(t) - a_1\sqrt{h_1(t)} - sgn(h_1(t) - h_2(t)) \cdot a_v x_i(t)\sqrt{|h_1(t) - h_2(t)|} = f_1(h_1, h_2, x_i), \tag{22}$$

$$A_2\rho\dot{h_2}(t) = sgn(h_1(t) - h_2(t)) \cdot a_v \rho x_i(t)\sqrt{|h_1(t) - h_2(t)|} - a_2\sqrt{h_2(t)} = f_2(h_1, h_2, x_i), \tag{23}$$

- determining all steady state values

$$x_{i0} = 1, h_{10} = 9.5111, h_{20} = 0.3658, \tag{24}$$

- calculating partial derivatives

$$\left.\frac{\delta f_1}{\delta h_1}\right|_0 = -a_1\frac{1}{2\sqrt{h_{10}}} - a_v\frac{1}{2\sqrt{h_{10} - h_{20}}}x_{i0} = -1.5827 = k_1, \tag{25}$$

$$\left.\frac{\delta f_1}{\delta h_2}\right|_0 = a_v\frac{1}{2\sqrt{h_{10} - h_{20}}}x_{i0} = 0.1465 = k_2, \tag{26}$$

$$\left.\frac{\delta f_1}{\delta x_i}\right|_0 = -a_v\sqrt{h_{10} - h_{20}} = -13.3952 = k_3, \tag{27}$$

$$\left.\frac{\delta f_2}{\delta h_1}\right|_0 = a_v\frac{1}{2\sqrt{h_{10} - h_{20}}}x_{i0} = 0.1465 = k_4, \tag{28}$$

$$\left.\frac{\delta f_2}{\delta h_2}\right|_0 = -a_v\frac{1}{2\sqrt{h_{10} - h_{20}}}x_{i0} - a_2\frac{1}{2\sqrt{h_{20}}} = -3.8082 = k_5, \tag{29}$$

$$\left.\frac{\delta f_2}{\delta \dot{y}}\right|_0 = a_v\sqrt{h_{10} - h_{20}} = 13.3952 = k_6, \tag{30}$$

- using Taylor series to determine linear equation

$$k_{11}\Delta\dot{h_1}(t) = k_1\Delta h_1(t) + k_2\Delta h_1(t) + k_3\Delta x_i(t), \tag{31}$$

$$k_{22}\Delta\dot{h_2}(t) = k_4\Delta h_1(t) + k_5\Delta h_1(t) + k_6\Delta x_i(t), \tag{32}$$

  where $k_{11} = A_1\rho$, $k_{22} = A_2\rho$

The state variables of the model are Δh_1 and Δh_2, while the input is Δx_i.

2.3 Transfer function

The next step is to determine the transfer function $G(s) = \frac{H_1(s)}{X_i(s)}$, where $H_1(s) = \mathcal{L}(\Delta h_1(t))$ and $X_i(s) = \mathcal{L}(\Delta x_i(t))$. If Laplace tranform is applied on equations [31][32], we obtain:

$$k_{11} s H_1 = k_1 H_1 + k_2 H_2 + k_3 X_i, \tag{33}$$

$$k_{22} s H_2 = k_4 H_1 + k_5 H_2 + k_6 X_i. \tag{34}$$

After some tranformations:

$$G(s) = \frac{k_{22} k_3 s + k_2 k_6 - k_3 k_5}{k_{11} k_{22} s^2 + (-k_{11} k_5 - k_{22} k_1) s + k_1 k_5 - k_2 k_4} = \frac{-40186 s - 49.05}{6000000 s^2 + 12365 s + 6.0059}. \tag{35}$$

2.4 Analysis of the fluid height difference

To determine the fluid height difference in the first tank in steady state between the linearized mathematical model and the nonlinear mathematical model for a step input signal $\Delta x_i(t) = 0.05 S(t)$, we can:

- calculate $h_1 0$ for $x_{i0} = 0.25$ for the nonlinear system

$$h_{10} = \frac{q_i^2}{a_1^2 + 2 a_1 a_v x_{i0} \sqrt{1 - b} + a_v^2 x_{i0}^2 (1 - b)} = 9.1215, \tag{36}$$

  where $b = \frac{a_v^2 x_{i0}^2}{a_2^2 + a_v^2 x_{i0}^2}$,

- calculate $h_{10} + \Delta h_{10}$ for $\Delta x_{i0} = 0.05$ for the linearized system (after obtaining the static equation)

$$\Delta h_{10} = \Delta x_{i0} \frac{k_2 k_6 - k_3 k_5}{k_1 k_5 - k_2 k_4} = -0.4083, \tag{37}$$

$$h_{10} + \Delta h_{10} = 9.5111 - 0.4083 = 9.1028, \tag{38}$$

- calculate the difference between the obtained values

$$h_{nonlinear} - h_{linearized} = 0.0187 m. \tag{39}$$

2.5 Simulink models

On the next page, developed Simulink models for nonlinear and linearized system can be found.

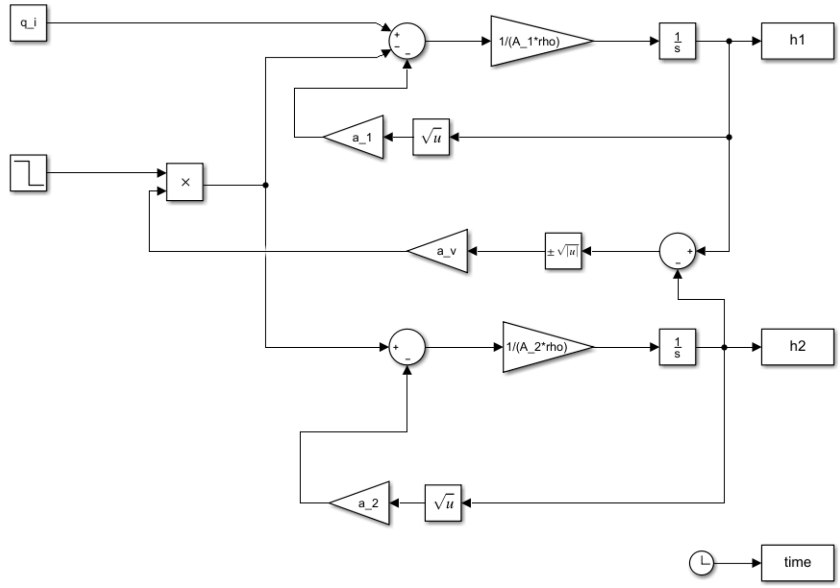


Figure 7: Simulink model for nonlinear system.

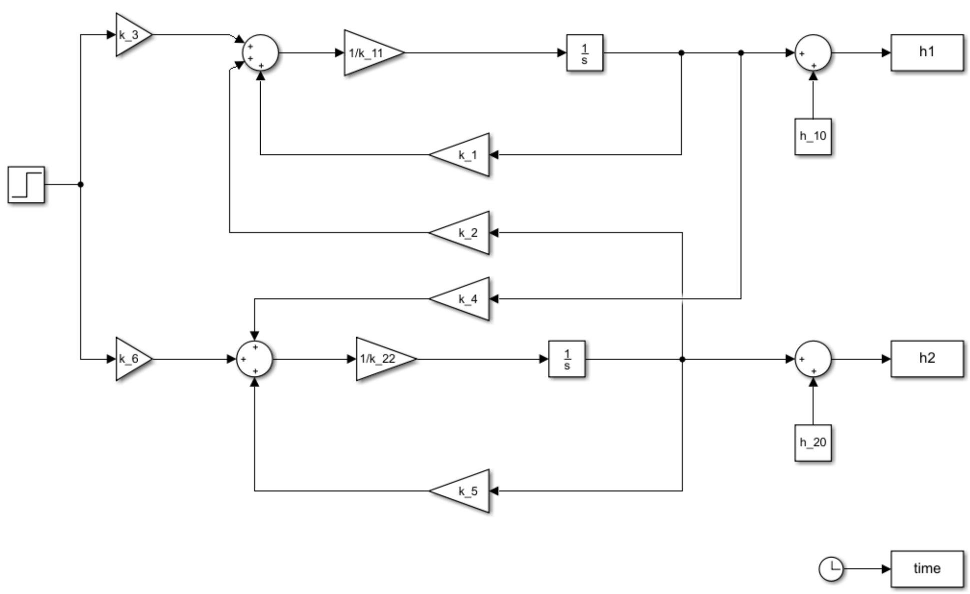


Figure 8: Simulink model for linearized system.

2.6 Simulations

On the figure below, the response of the fluid height in the first tank for both linearized and nonlinear models with the input signal $\Delta x_i(t) = 0.05S(t$ - 200) has been plotted.

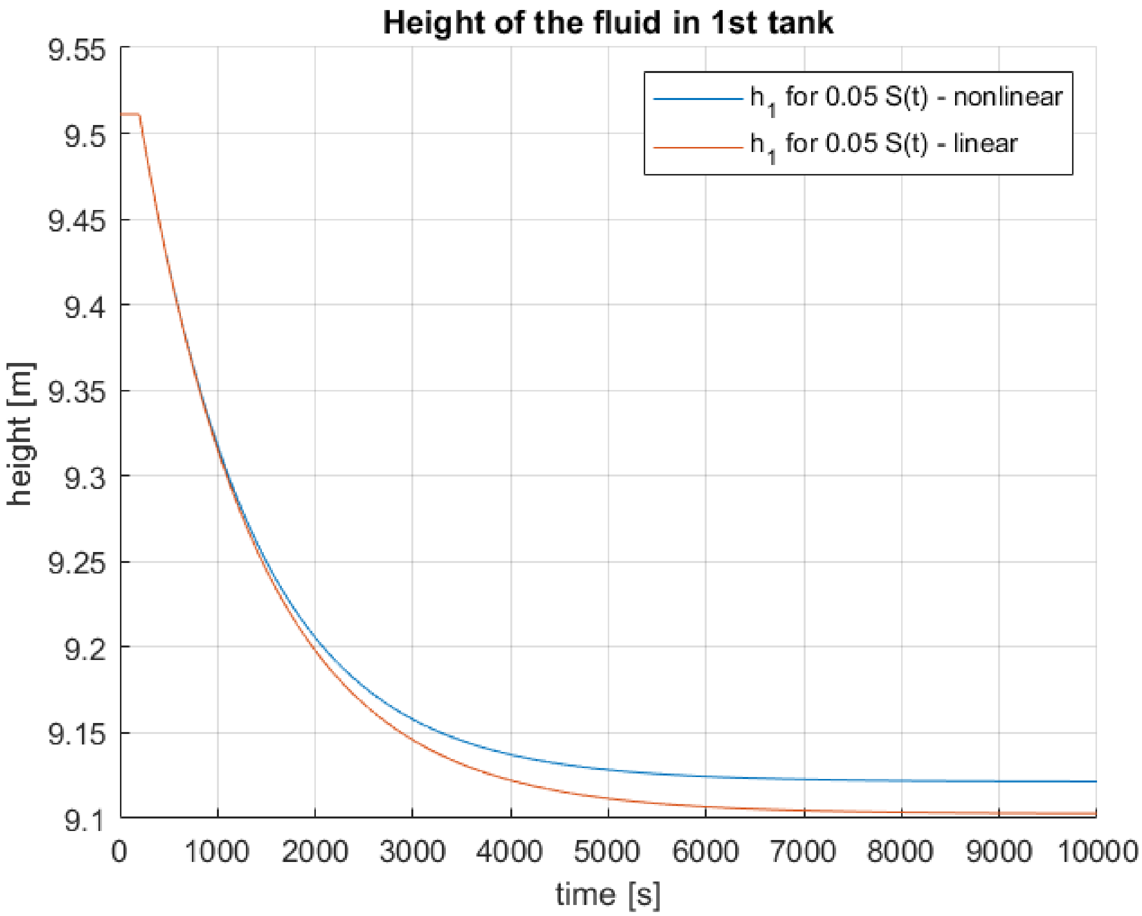


Figure 9: Comparison of systems' responses.

Below the steady state difference between responses of linearized and nonlinear model is plotted. It can be seen that it is consistent with the analytical result obtained in point 2.4.

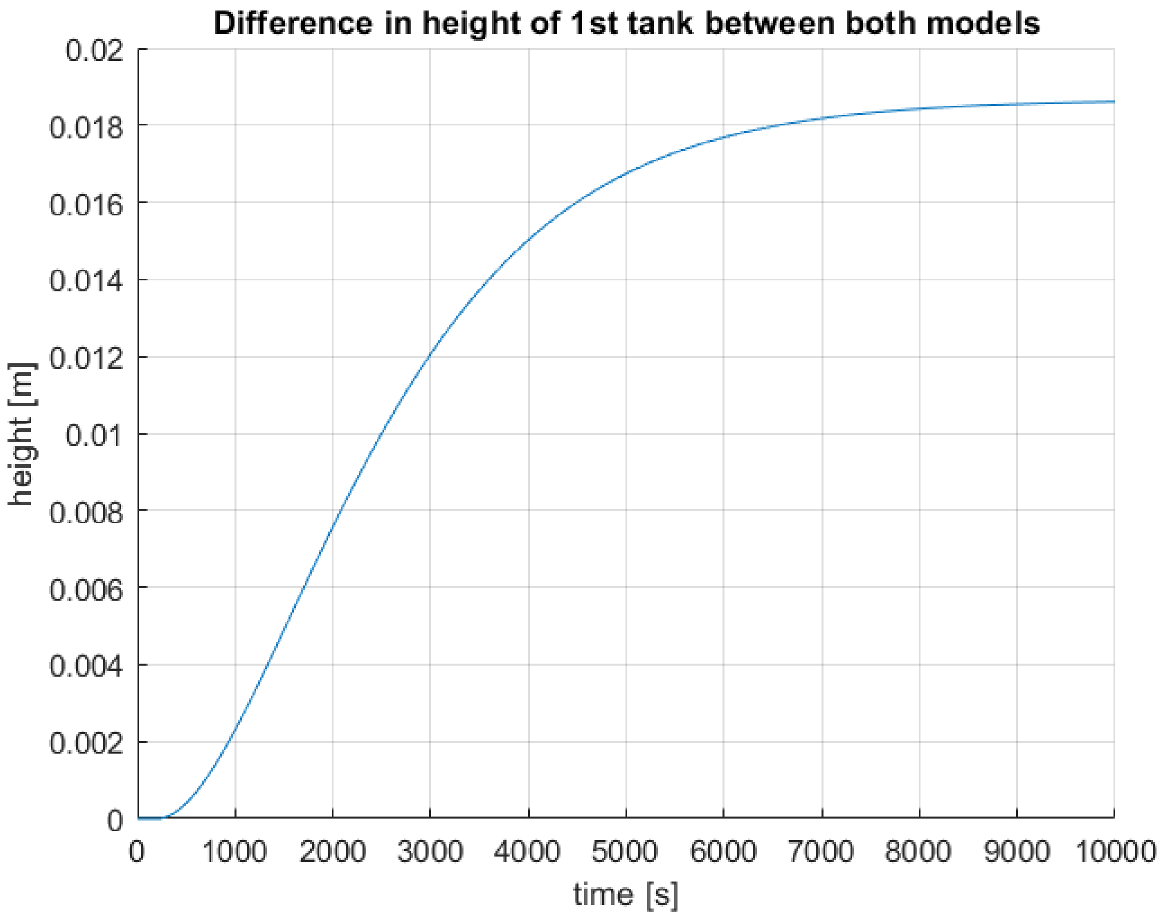


Figure 10: Plotted difference.