FREQUENCY-DOMAIN ANALYSIS
3^{rd} homework

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11.11.2020r.

1 Assignment [1]

A system is described by the following transfer function:

$$G(s) = 10\frac{s+1}{(s+2)(s+5)} = \frac{10(s+1)}{s^2+7s+10} \tag{1}$$

1.1 Bode diagram using the asymptote approximation

To obtain a Bode diagram, it is needed to transform the transfer function in the following way:

$$G(s) = 10\frac{s+1}{(s+2)(s+5)} = \frac{10(s+1)}{2(\frac{1}{2}s+1)5(\frac{1}{5}s+1)} = \frac{s+1}{(\frac{1}{2}s+1)(\frac{1}{5}s+1)} \tag{2}$$

It can be seen that for $\omega = 2$ and $\omega = 5$ the slope will change $-20dB/decade$, while for $\omega = 1$ the slope will raise $+20dB/decade$. Regarding the phase, changes of +45 deg/decade will be seen at point $\omega = 0.1$, $\omega = 20$, $\omega = 50$, -45 deg/decade at points $\omega = 0.2$, $\omega = 0.5$ and $\omega = 10$. The asymptote approximation is marked on the Bode diagram in the next subsection,

1.2 Bode diagram using MATLAB bode() function

To check the appoximation by asymptotes, a Bode diagram using bode() function has been superimposed on the drawing from the previous task.

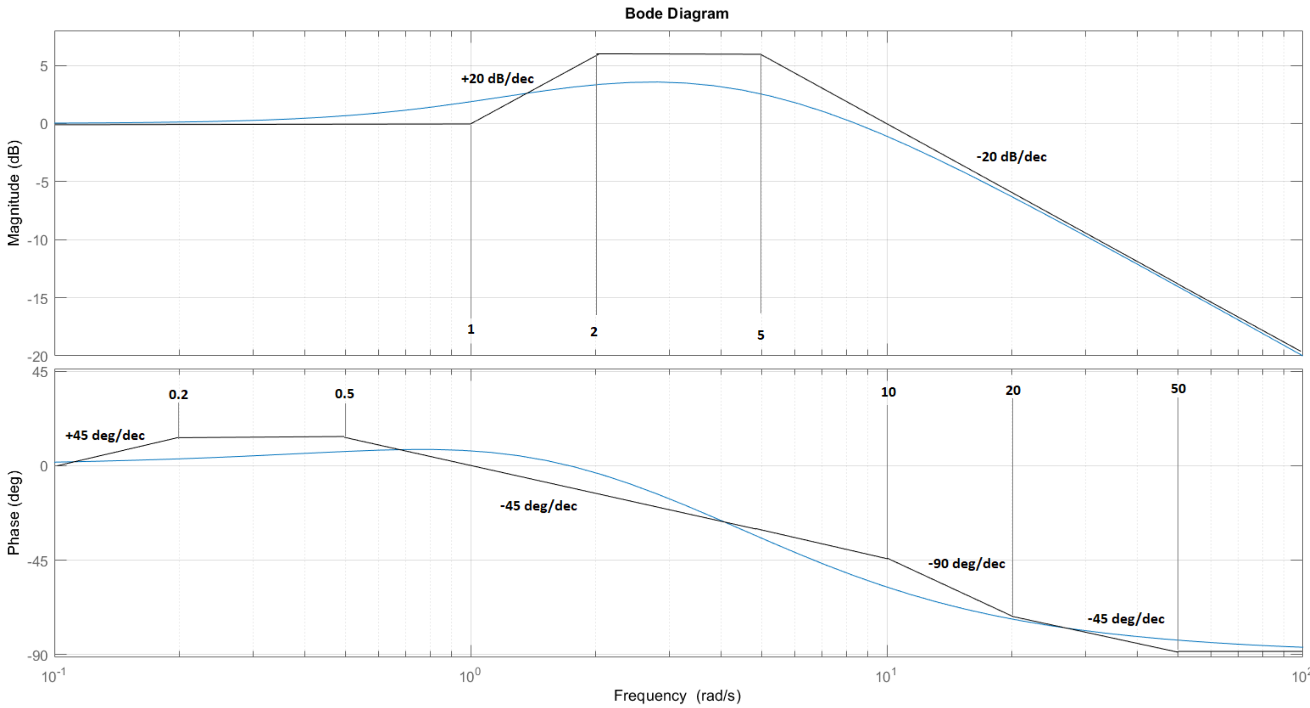

Figure 1: System's approximation and accurate solution.

The biggest deviations are at the break frequencies - it is because our approximations are made on the assumptions that $\omega << \omega_b$ or $\omega >> \omega_b$. So the further from ω_r we are, the better our approximation is and the other way round - the closer we are, the worse it gets.

1.3 Manual Nyquist diagram

For the purpose of obtaining the Nyquist diagram, $G(j\omega)$ is computed and broken down on its real and imaginary components:

$$G(j\omega) = \frac{10j\omega + 10}{(j\omega)^2 + 7j\omega + 10} = \frac{10j\omega + 10}{7j\omega + (10 - \omega^2)} = \frac{(10j\omega + 10)(7j\omega - (10 - \omega^2))}{-49\omega^2 - (10 - \omega^2)^2} = \frac{60\omega^2 + 100 + j\omega(30 - 10\omega^2)}{\omega^4 + 29\omega^2 + 100} \tag{3}$$

$$Re\{G(j\omega)\} = \frac{60\omega^2 + 100}{\omega^4 + 29\omega^2 + 100} \tag{4}$$

$$Im\{G(j\omega)\} = \frac{\omega(30 - 10\omega^2)}{\omega^4 + 29\omega^2 + 100} \tag{5}$$

To plot the Nyquist diagram, we need to find the points of intersection with each axis. To do that, imaginary and real part of $G(j\omega)$ need to be equal to 0:

$$Re\{G(j\omega)\} = 0 \Leftrightarrow \omega \in \emptyset, \tag{6}$$

$$Im\{G(j\omega)\} = 0 \Leftrightarrow \omega = 0,\ \omega = \sqrt{3}. \tag{7}$$

For $\omega = 0 \rightarrow G(j\omega) = 1$, $\omega = \sqrt{3} \rightarrow G(j\omega) = \frac{280}{196} \approx 1.43$. The graph never goes to the left side of complex plane (it only goes through the I and IV quadrant). If we take $\omega \in [0; \sqrt{3}]$, real part will be positive, so the line joining these two omegas can be drawn in the first quadrant. For $\omega \rightarrow \infty$, $G(j\omega) \rightarrow 0$, so the final point will be in the origin (through the IV quadrant). The initial angle is equal to 0^o, the final to 270^o.

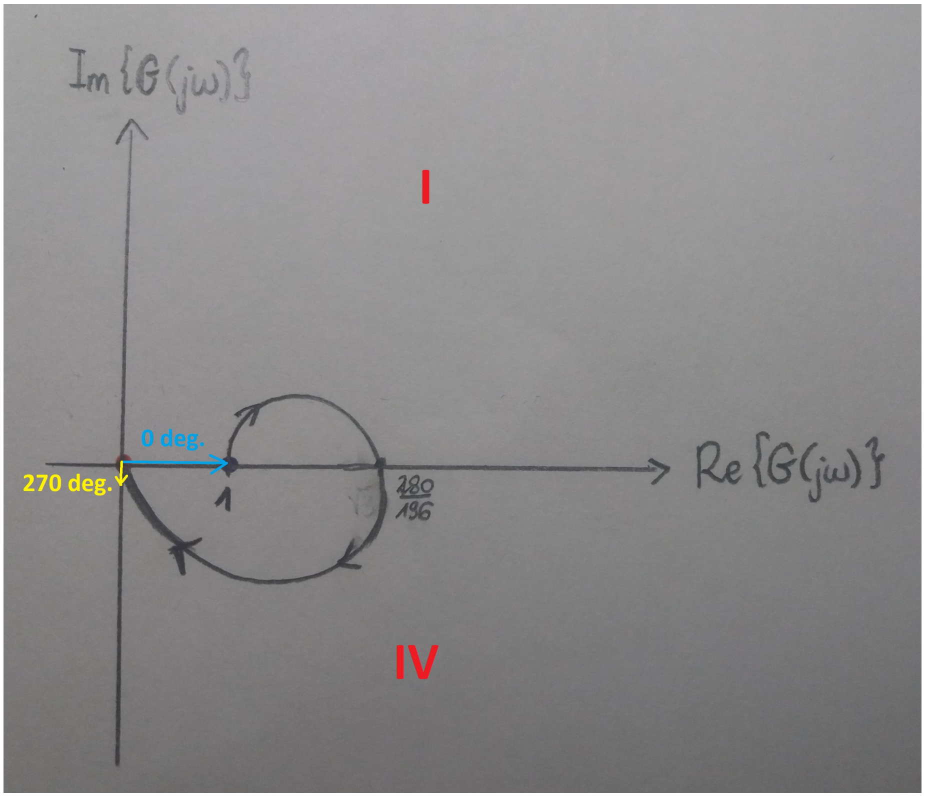

Figure 2: Nyquist diagram - manual version.

1.4 Nyquist diagram

By running the special MATLAB function, the Nyquist diagram of the system looks like in the picture below:

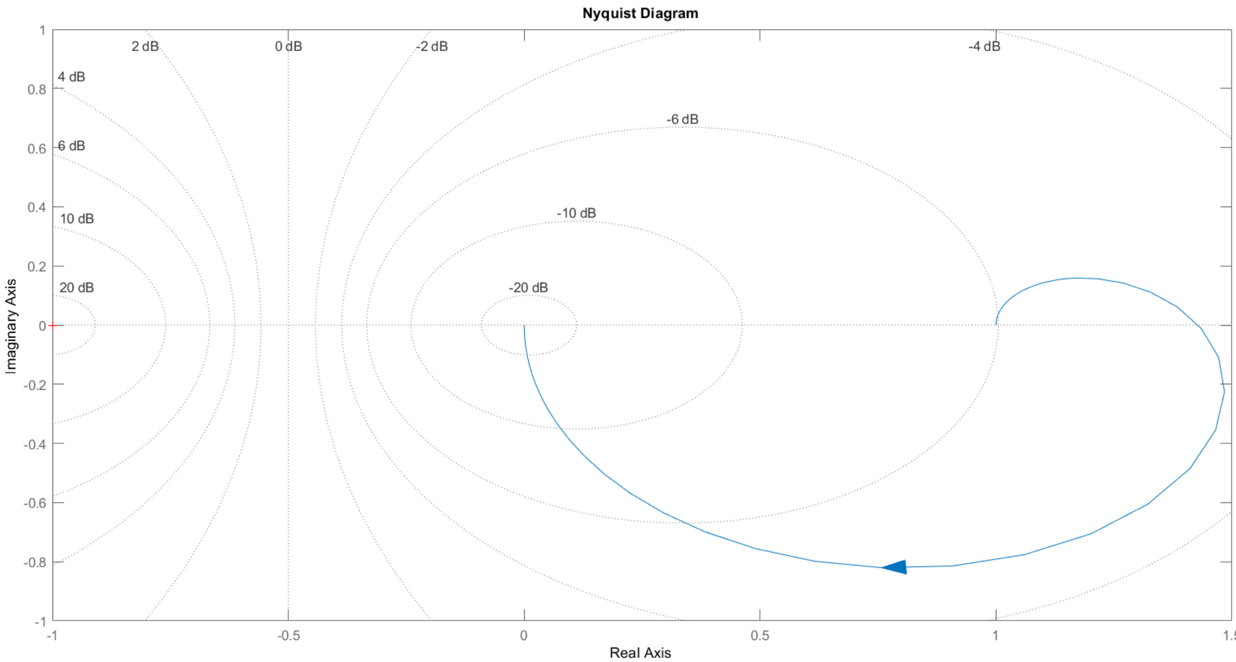

Figure 3: Nyquist diagram - MATLAB version.

1.5 Analytical computation of the output

The system is excited by the following harmonic input signal: u(t) = 3 sin(5t). To compute the amplitude A_m, the frequency ω_0 and the phase delay $\phi(\omega_0)$ of the output signal ($y(t) = A_m sin(\omega_0 t + \varphi(\omega_0))$) in the steady state, $|G(j\omega)|$ needs to be calculated:

$$|G(j\omega)| = \sqrt{\frac{(60\omega^2 + 100)^2 + (\omega(30 - 10\omega^2))^2}{(\omega^4 + 29\omega^2 + 100)^2}} = \frac{\sqrt{100\omega^6 + 3000\omega^4 + 12900\omega^2 + 10000}}{\omega^4 + 29\omega^2 + 100}. \tag{8}$$

The frequency ω doesn't change, so $\omega_0 = \omega = 5$. The amplitude can be computed in the following way:

$$A_m = A_0|G(j\omega_0)| = 3\frac{\sqrt{100 \cdot 5^6 + 3000 \cdot 5^4 + 12900 \cdot 5^2 + 10000}}{\cdot 5^4 + 29 \cdot 5^2 + 100} = 4.0172. \tag{9}$$

The phase delay is the result of arc tangent of the $Im(G(j\omega_0)$ to $Re(G(j\omega_0)$ ratio. Therefore:

$$\varphi(\omega_0) = arctan\frac{Im(G(j\omega_0)}{Re(G(j\omega_0)} = arctan\frac{\omega_0(30 - 10\omega_0^2)}{60\omega_0^2 + 100} = arctan\frac{5(30 - 10 \cdot 5^2)}{60 \cdot 5^2 + 100} \approx -0.6023. \tag{10}$$

Finally, $y(t) = 4.0172sin(5t - 0.6023)$.

1.6 Marking the values

$G(j\omega_0)$, $|G(j\omega_0)|$ and $arg[G(j\omega_0)]$ for the frequency ω_0 obtained in the previous task have been marked onn the Nyquist diagram:

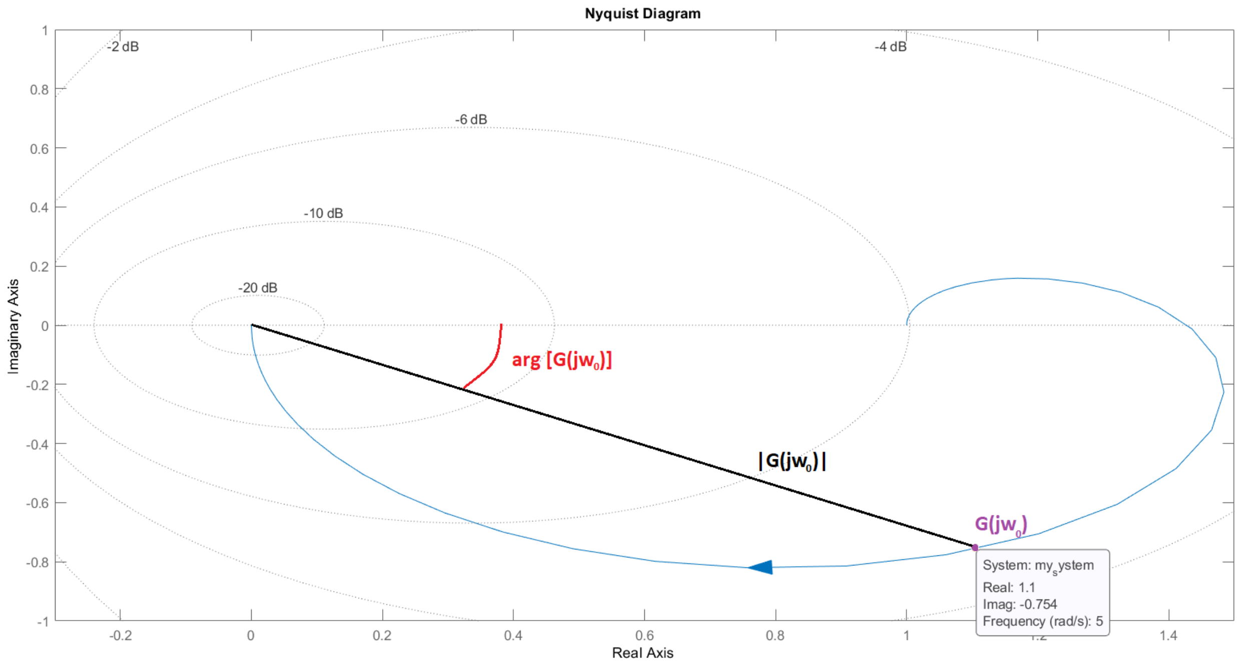

Figure 4: Marked points.

1.7 Caclulating the error

Using the linear equations of the asymptotes from the Bode approximation, we can compute, that:

$$|G(j5)| = 0 + 20log\frac{2}{1} + 0 = 20log2, \tag{11}$$

$$arg[G(j5)] = 45log\frac{0.2}{0.1} + 0 - 45log\frac{5}{0.5} = 45log2 - 45, \tag{12}$$

On this basis, the error obtained from this approximate approach with respect to the exact values can be calculated:

$$error(|G(j5)|) = 20log2 - 4.0172 \approx 2.00 \text{ dB}, \tag{13}$$

$$error(arg[G(j5)]) = -0.6023 - (45log2 - 45) \cdot \pi/180 \approx -0.0533 \text{ rad}. \tag{14}$$

1.8 Simulink solution

In order to check the obtained solution in output computation subsection, the following Simulink model has been created:

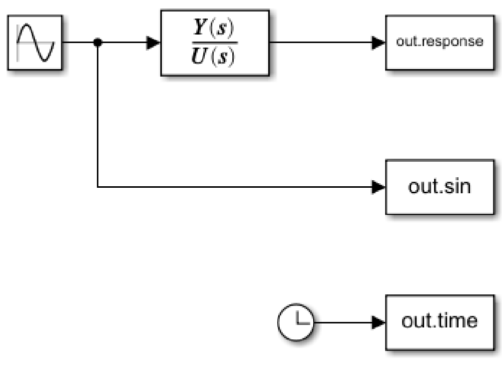

Figure 5: Simulink model.

The input sine wave, the output and the calculated sine wave have been plotted:

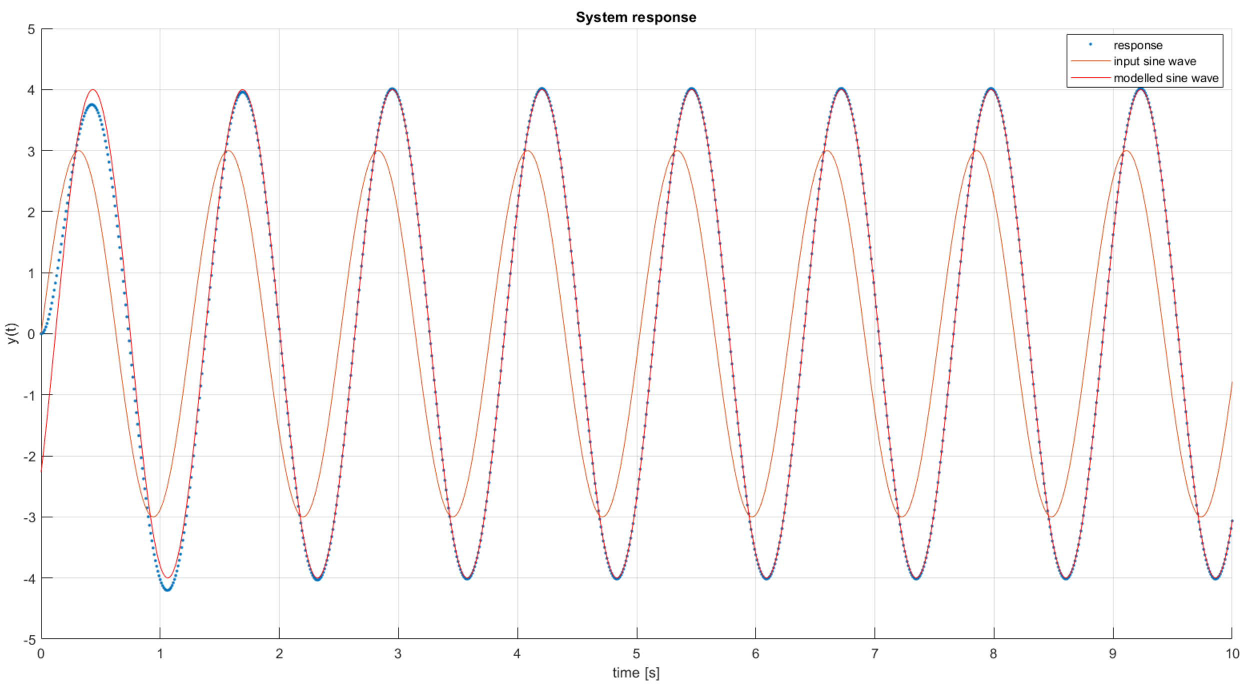

Figure 6: Simulation result.

It can be seen that the modelled response and the actual response match. It is impossible to check the obtained parameters of the signal (amplitude, phase delay, frequency) already in the first period because at the beginning the system's output is in the transition state - we need to wait some time before we can obtain these parameters from the steady state.

2 Assignment [2]

A system is described by the following transfer function:

$$G(s) = \frac{1}{s^2 + 0.6s + 1} \tag{15}$$

2.1 Bode diagram using the asymptote approximation

It case of the above transfer function, till $\omega_n = 1$ the slope is equal to zero (is on the level of 20 log K, in this case equal to 0), and after achieving the value of ω_n the slope decreases -40dB/decade.

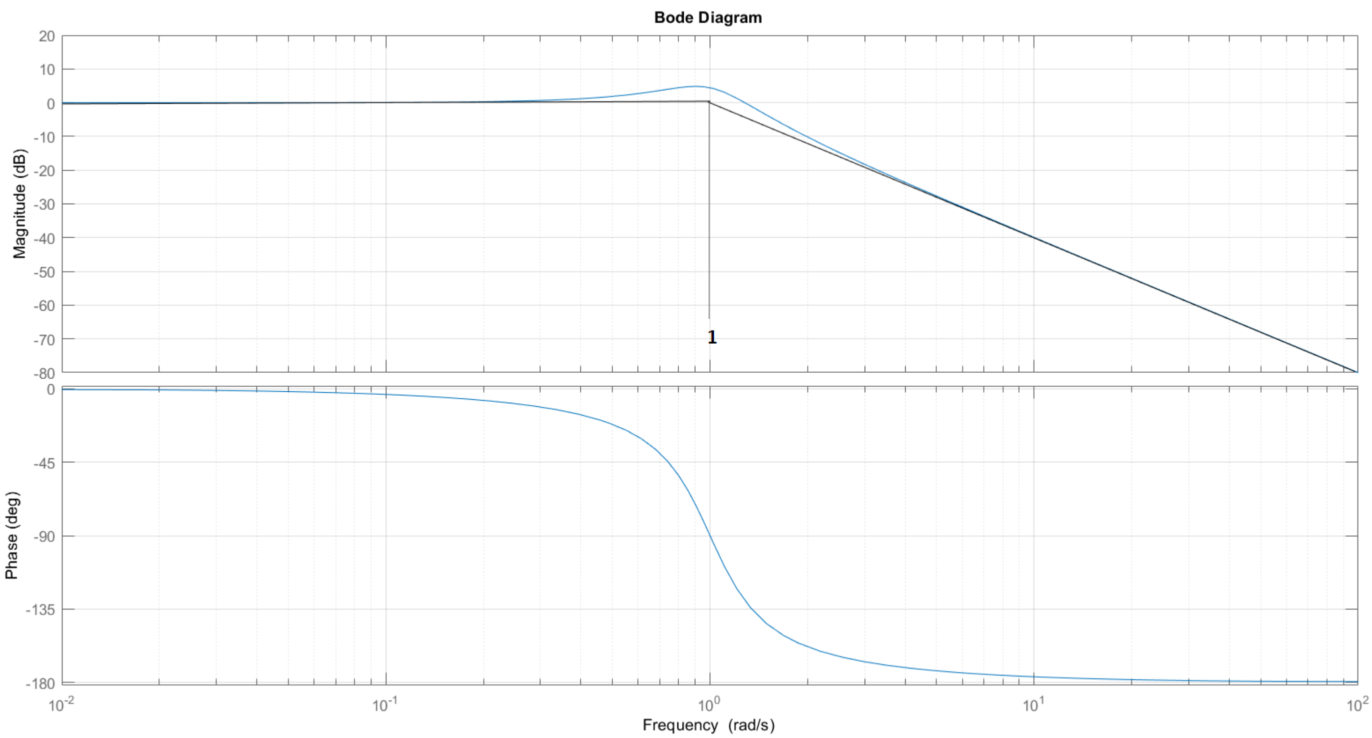

Figure 7: Bode diagram of the system.

2.2 Determining parameters

- Natural frequency ω_n:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{16}$$

$$\omega_n^2 = 1 \longrightarrow \omega_n = 1 \text{ (assumption: } \omega_n > 0) \tag{17}$$

- Gain K:

$$K\omega_n^2 = 1 \longrightarrow K = 1 \tag{18}$$

- Relative damping coefficent ξ:

$$2\xi\omega_n = 0.6 \longrightarrow \xi = 0.3 \tag{19}$$

- Maximum value of the amplitude-frequency characteristic $A(\omega)$:

$$A(\omega)_{max} = \frac{K}{2\xi\sqrt{1-\xi^2}} = \frac{1}{0.6\sqrt{1-0.09}} = \frac{1}{0.54} \approx 1.852 \tag{20}$$

- Resonant frequency ω_r:

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = \sqrt{0.82} \approx 0.906 \tag{21}$$

2.3 Damping coefficient's impact on the system

Eight more transfer functions $G_i(s) = \frac{K\omega_n^2}{s^2+2\xi_i\omega_n s+\omega_n^2}$ with the same gain K and the same natural frequency ω_n as the original system are constructed. The only variable here is the damping coefficent ξ. We obtain the following systems:

- $\xi = 0.005$: $G_1(s) = \frac{1}{s^2+0.01s+1}$,
- $\xi = 0.05$: $G_2(s) = \frac{1}{s^2+0.1s+1}$,
- $\xi = 0.125$: $G_3(s) = \frac{1}{s^2+0.25s+1}$,
- $\xi = 0.5$: $G_4(s) = \frac{1}{s^2+s+1}$,
- $\xi = 1$: $G_5(s) = \frac{1}{s^2+2s+1}$,
- $\xi = 1.5$: $G_6(s) = \frac{1}{s^2+3s+1}$,
- $\xi = 2$: $G_7(s) = \frac{1}{s^2+4s+1}$,
- $\xi = 2.5$: $G_8(s) = \frac{1}{s^2+5s+1}$.

All Bode diagrams of all nine systems (the original one and 8 additional) are plotted on the same figure.

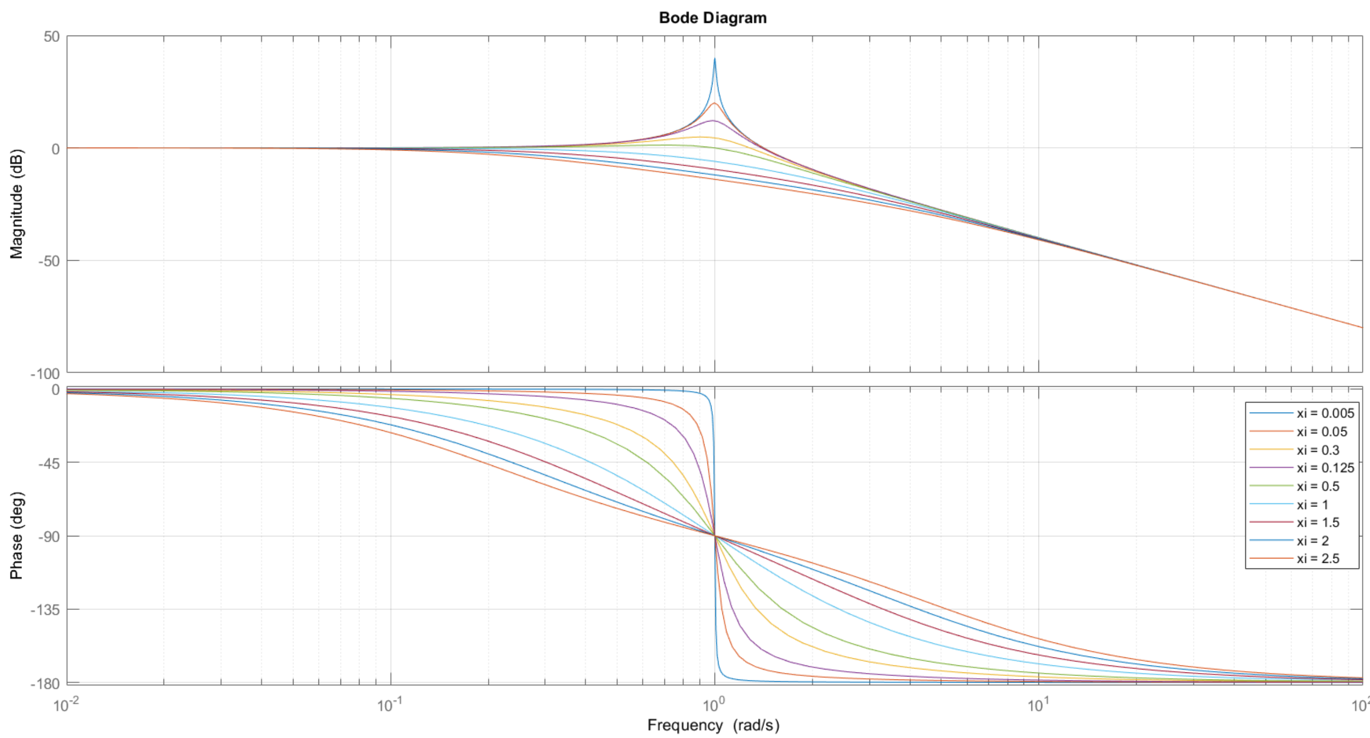

Figure 8: Damping coefficient's impact on the system.

The closer ξ is to zero, the worst our approximation with the asymptotes is. For $\xi \longrightarrow 0$, we can observe that $A(\omega)_{max} \longrightarrow \infty$. This can be also concluded from the formula of $A(\omega)_{max}$.