TIME-DOMAIN ANALYSIS
4^{th} homework

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1 Assignment [1]

A system is described by the following state-space model:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \tag{1}$$

$$y = \mathbf{C}x + \mathbf{D}u \tag{2}$$

where:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -18 & -9 \end{bmatrix} \tag{3}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4}$$

$$\mathbf{C} = 18 \begin{bmatrix} a & 1 \end{bmatrix} \tag{5}$$

$$\mathbf{D} = \begin{bmatrix} 0 \end{bmatrix} \tag{6}$$

1.1 Step and impulse response functions

To determine the step and impulse response functions, the transfer function needs to be determined first. After applying Laplace transform on the equation and after transforming the equation, the following dependence is obtained:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{7}$$

If we put our values to the equation:

$$\mathbf{G}(s) = 18 \begin{bmatrix} a & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -18 & -9 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \tag{8}$$

We need to do further transformations to obtain the proper form of transfer function:

$$\mathbf{G}(s) = 18 \begin{bmatrix} a & 1 \end{bmatrix} \cdot \begin{bmatrix} s & -1 \\ 18 & s+9 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

$$\mathbf{G}(s) = 18 \begin{bmatrix} a & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{s+9}{(s+3)(s+6)} & \frac{1}{(s+3)(s+6)} \\ \frac{-18}{(s+3)(s+6)} & \frac{s}{(s+3)(s+6)} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10}$$

$$\mathbf{G}(s) = 18 \begin{bmatrix} a & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{(s+3)(s+6)} \\ \frac{s}{(s+3)(s+6)} \end{bmatrix} \tag{11}$$

$$\mathbf{G}(s) = \begin{bmatrix} \frac{18(a+s)}{(s+3)(s+6)} \end{bmatrix} \tag{12}$$

It is known that to calculate the output of the system with the use of transfer function, we use the dependence $Y(s) = G(s)U(s)$. For step function input $U(s) = \frac{1}{s}$, while for impulse function input $U(s) = 1$. Therefore:

$$h(t) = \mathcal{L}\left\{ \frac{18(a+s)}{s(s+3)(s+6)} \right\} = (6-2a)e^{-3t} + (a-6)e^{-6t} + a \tag{13}$$

$$g(t) = \mathcal{L}\left\{ \frac{18(a+s)}{(s+3)(s+6)} \right\} = 6(6-a)e^{-6t} + 6(a-3)e^{-3t} \tag{14}$$

1.2 Lack of natural mode of the system

Individual natural modes of the system do not show in h(t) when a = 3 or a = 6. For such values of the parameter coefficients next to exponential functions in step response function equal to zero and therefore individual natural modes of the system do not show.

1.3 Influence of the parameter a

To see how the parameter a infuence the steady state of a step response function, we need to calculate how the response looks like while striving to infinity:

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} ((6-2a)e^{-3t} + (a-6)e^{-6t} + a) = a \tag{15}$$

It can be seen that the system in steady state is directly dependent on the value of a.
For the impulse response function at time instant $t = 0^+$, we can calculate that:

$$\lim_{t\to 0^+} g(t) = \lim_{t\to\infty} sG(s) = \lim_{t\to\infty} \frac{18s(a+s)}{(s+3)(s+6)} = 18 \tag{16}$$

In this case, the parameter a doesn't influence the given situation.
Depending on the parameter, we obtain different shapes of the step response function. If a=3 or a=6, then one of the

exponential components disappears. The parameter a influences on which level and with what manner the response stabilizes. Beneath, there is a plot of the responses for different values of a.

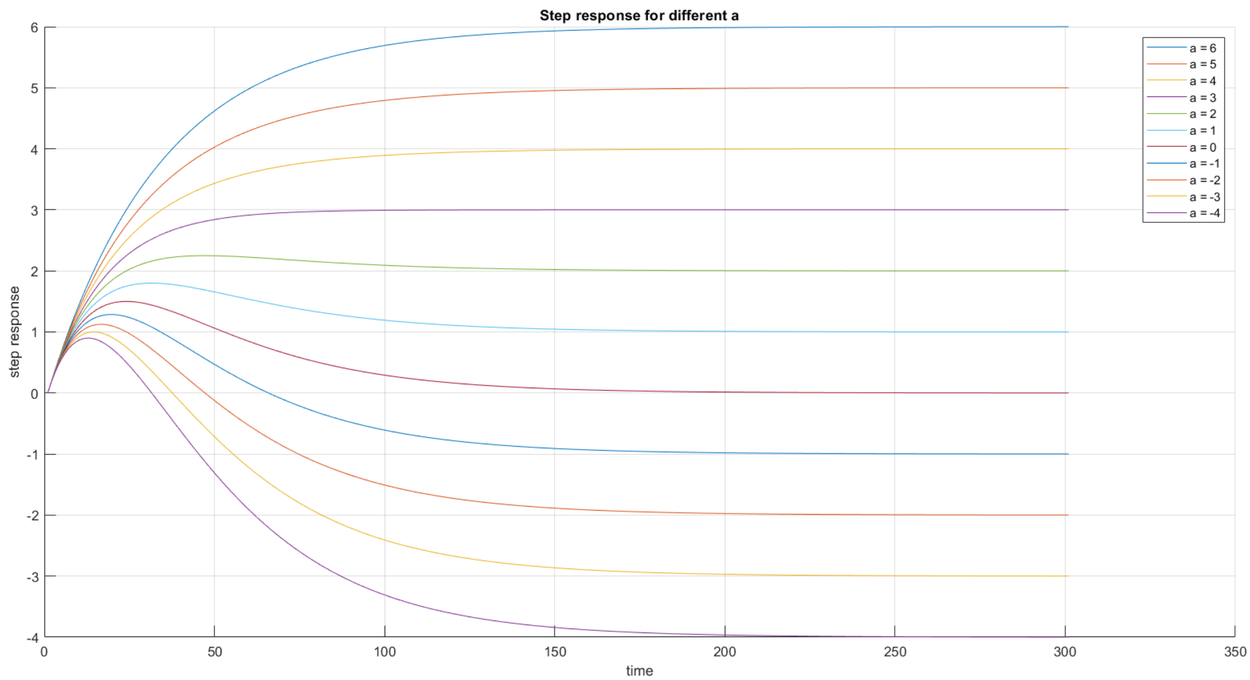

Figure 1: Step response for different parameters

1.4 Overshoot

The step response function has the following form:

$$h(t) = (6 - 2a)e^{-3t} + (a - 6)e^{-6t} + a. \tag{17}$$

The range of values of $a > 0$ for which the h(t) has an overshoot needs to be determined. The step response function has an overshoot if it has a maximum for $t > 0$.

$$h'(t) = g(t) = 6(6 - a)e^{-6t} + 6(a - 3)e^{-3t} = 0 \tag{18}$$

$$6(6 - a)e^{-6t} + 6(a - 3)e^{-3t} = 0 \tag{19}$$

$$(6 - a)e^{-6t} + (a - 3)e^{-3t} = 0 \tag{20}$$

$$6 - a + (a - 3)e^{3t} = 0 \tag{21}$$

$$6 - a = (3 - a)e^{3t} \tag{22}$$

If a=3, then we lose the dependence on t, so we will only follow the case when $a \neq 3$.

$$\frac{6-a}{3-a} = e^{3t} \tag{23}$$

We have an additional assumption to satisfy this equality: $\frac{6-a}{3-a} > 0 \Longrightarrow a \in (0;3) \cup (6;\infty)$.

$$ln\frac{6-a}{3-a} = 3t \tag{24}$$

$$\frac{1}{3}ln\frac{6-a}{3-a} = t \tag{25}$$

From the condition $t > 0$:

$$\frac{1}{3}ln\frac{6-a}{3-a} > 0 \tag{26}$$

$$ln\frac{6-a}{3-a} > 0 \tag{27}$$

$$\frac{6-a}{3-a} > 1 \tag{28}$$

$$\frac{6-a}{3-a} - 1 > 0 \tag{29}$$

$$\frac{6-a-3+a}{3-a} > 0 \tag{30}$$

$$\frac{3}{3-a} > 0 \tag{31}$$

$$3 > a \tag{32}$$

After calculating the second derivative of h(t) we know that the found solution is the maximum (not the minimum) of the function. Therefore, concluding all the obtained assumptions, the overshoot appears if $a \in (0;3)$.

1.5 Poles and zeros

On the figures below, step response and the arrangement of system's poles and zeros in the complex s-plane for the case when there is an overshoot (e.g. a = 1), for the case when there isn't one (e.g. a = 4) and for the corner case (a = 3) are shown.

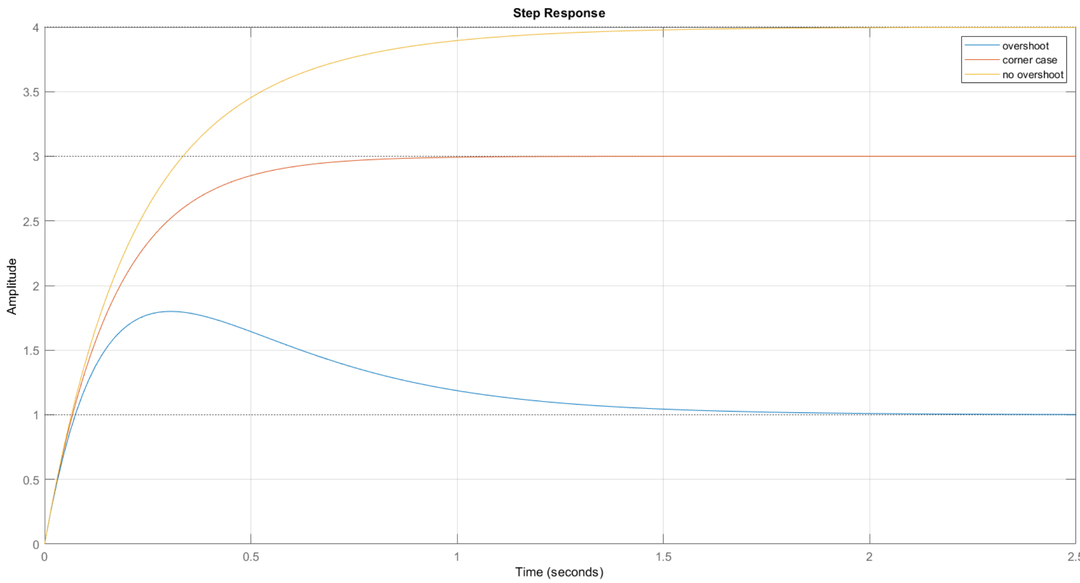

Figure 2: Step responses for three cases

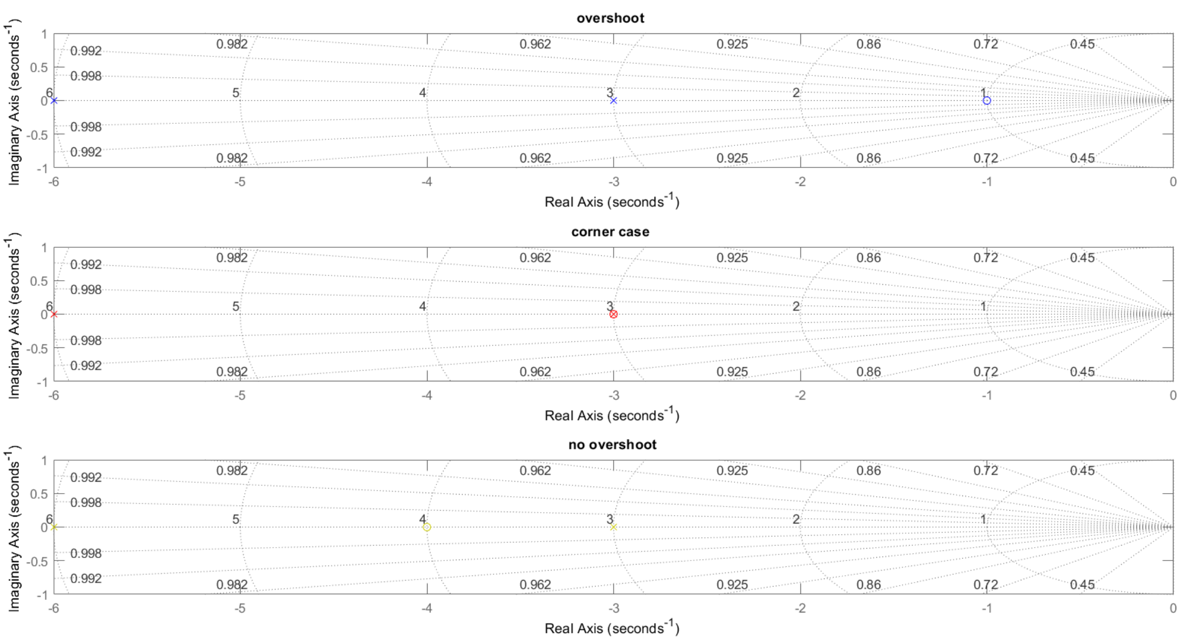

Figure 3: Pole-zero map for three cases

1.6 Negative parameter a

Using the Matlab function step, the step response function for the case when a = -1 has been generated. In the step response function a negative gain/a negative steady state can be seen. At the beginning the values move away from the steady state to get close to it later.

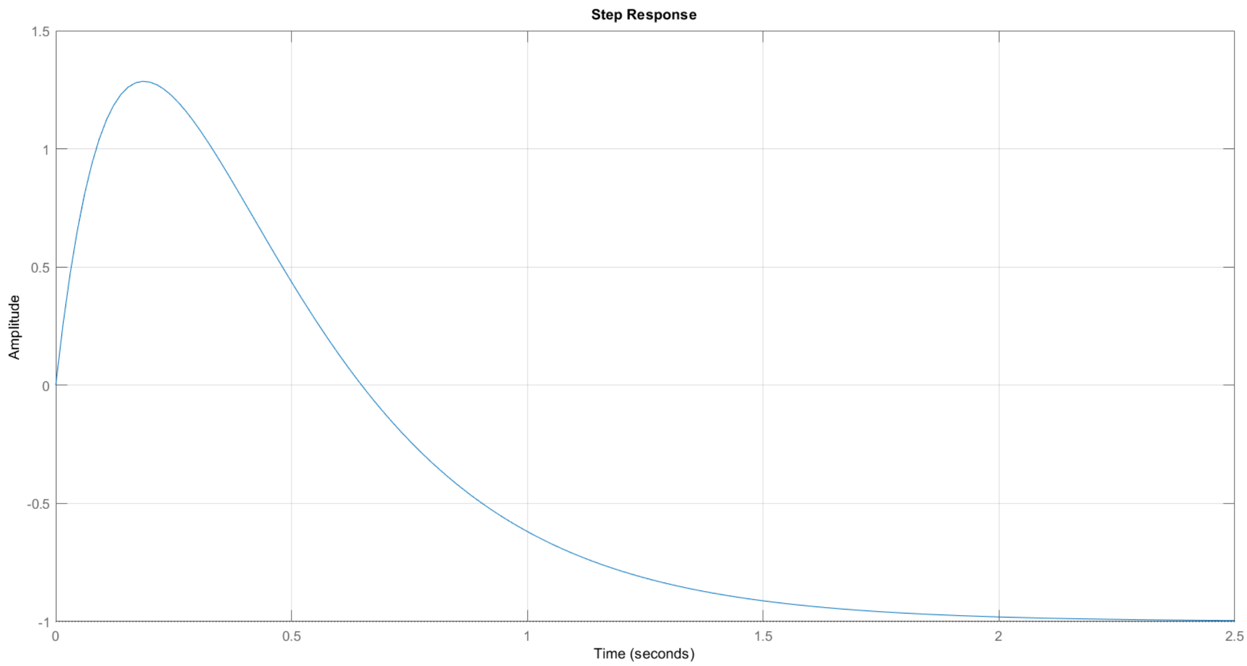

Figure 4: Step response for a=-1

The locations of poles and zeros in that case are shown below - zero is in the right half-plane, poles in the left half-plane (without imaginary part).

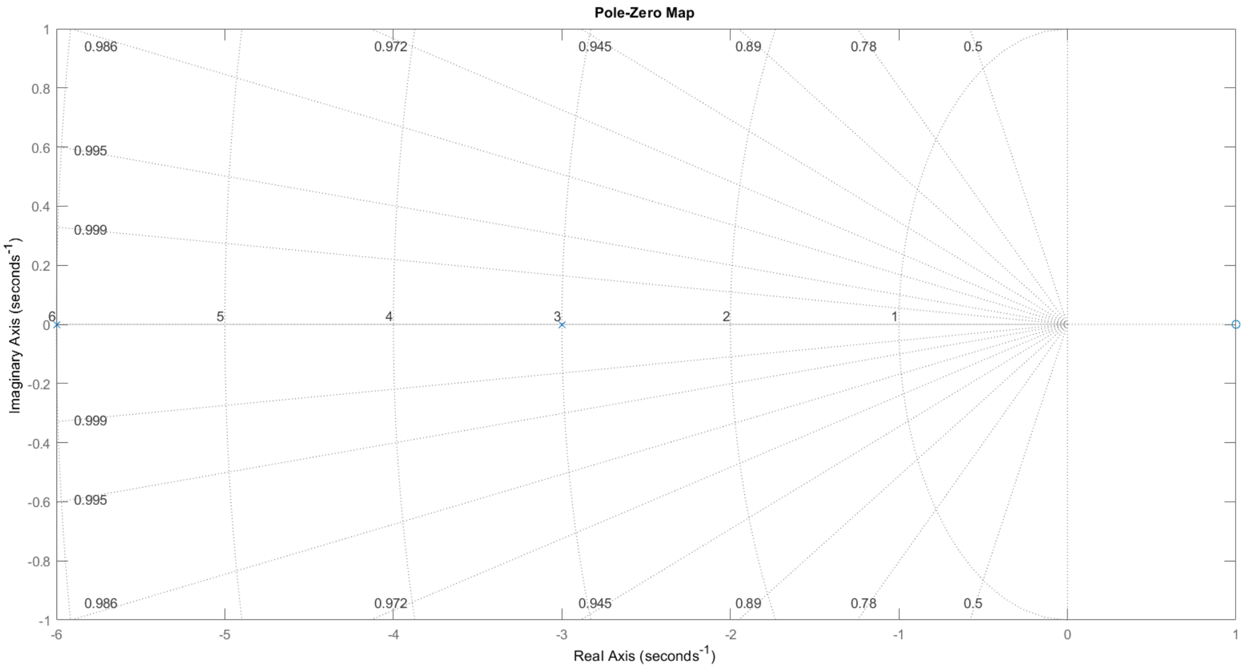

Figure 5: Pole-zero map for a=-1

2 Assignment [2]

A transfer function of an open-loop system with an I-element and a PT1-element has the following general form:

$$G_o(s) = \frac{K}{s(1 + Ts)}. \tag{33}$$

2.1 Closed-loop system

Determinng the transfer function $G_r(s)$ of a corresponding closed-loop system with a unit negative feedback loop looks like:

$$G_r(s) = \frac{G_o(s)}{G_o(s) + 1} = \frac{K}{K + s(1 + Ts)} = \frac{K}{Ts^2 + s + K}. \tag{34}$$

2.2 Open-loop vs closed-loop system's response

We consider the following particular values of the parameters: K = K_1 = 3 and T = T_1 = 1. The corresponding transfer functions of the open-loop and the closed-loop system are denoted by $G_{o,1}(s)$ and $G_{r,1}(s)$, respectively. These transfer functions are obtained by substituting K with K_1 and T with T_1. On the same figure the step response functions corresponding to $G_{o,1}(s)$ and $G_{r,1}(s)$ are plotted using the Matlab function step:

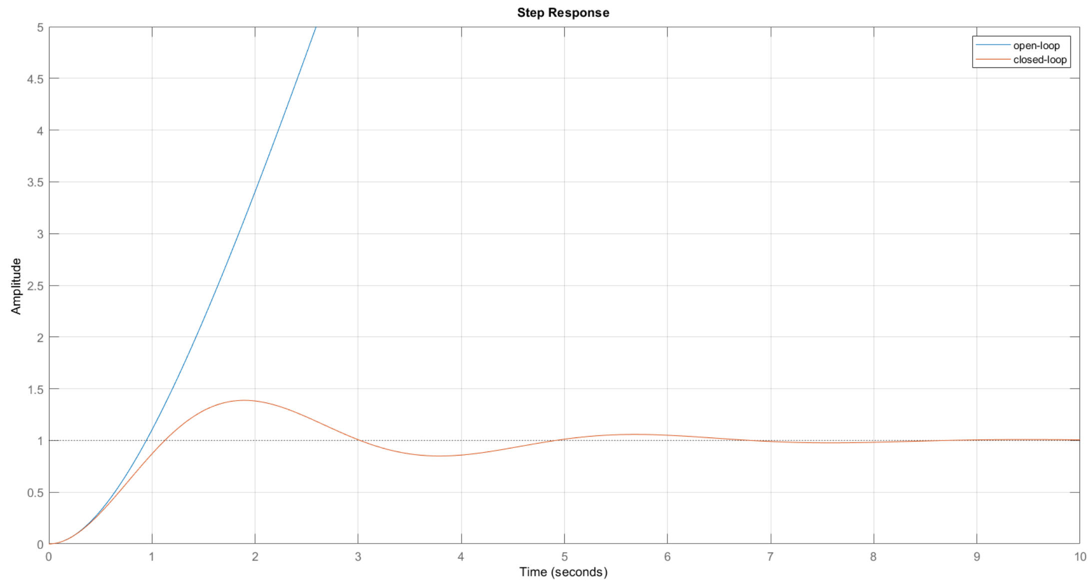

Figure 6: Step responses for open-loop and closed-loop systems.

The integral element in the transfer function can be visible in the open-loop system. It can be seen in the step response that the system doesn't have a steady state. The response for a wider range of time has been plotted to show that dependence.

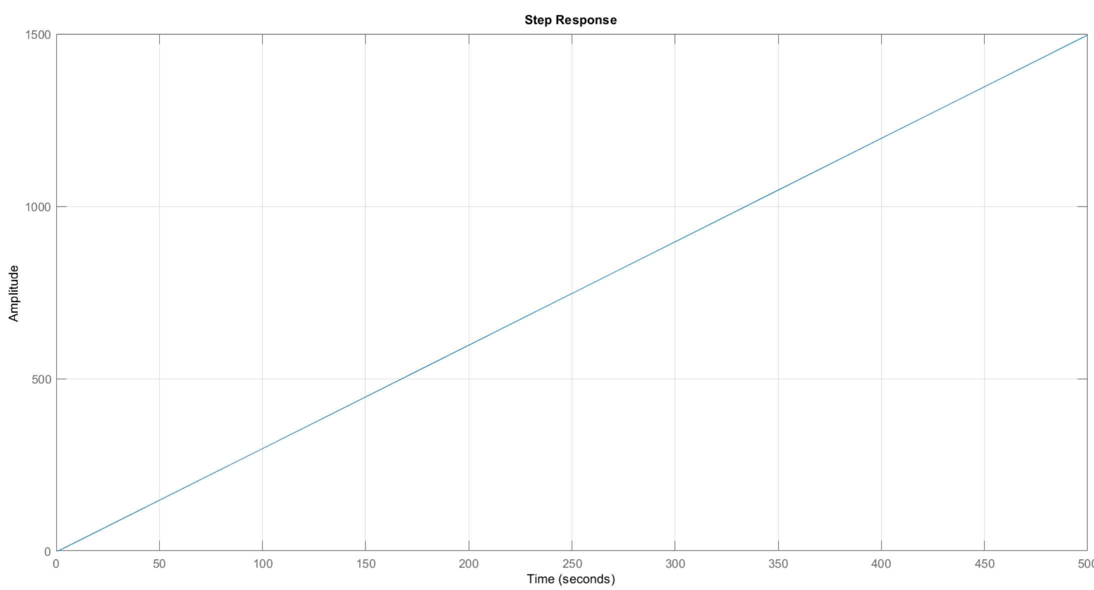

Figure 7: Step responses for the open-loop system.

2.3 Time-domain quality indicators

For the following transfer function:

$$G_{r,1}(s) = \frac{3}{s^2 + s + 3}. \tag{35}$$

time-domain quality indicators t_m, σ_m, $t_{1\%}$ and t_r need to be determined. First, ω_n and ξ have to be calculated. If we compare $G(s) = \frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$ with our transfer function, we can say that $\xi = \frac{\sqrt{3}}{6}$ and $\omega_n = \sqrt{3}$. Then:

$$t_m = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\sqrt{3}\sqrt{1-\left(\frac{\sqrt{3}}{6}\right)^2}} \approx 1.8945, \tag{36}$$

$$\sigma_m[\%] = 100e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} = 100e^{-\frac{\frac{\sqrt{3}}{6}\pi}{\sqrt{1-\left(\frac{\sqrt{3}}{6}\right)^2}}} \approx 38.7815\%, \tag{37}$$

$$t_{1\%} = \frac{4.6}{\omega_n\xi} = \frac{4.6}{\sqrt{3}\frac{\sqrt{3}}{6}} = 9.2, \tag{38}$$

$$t_r = \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{3}} \approx 1.0392. \tag{39}$$

2.4 Poles in the left half-plane

The general formula for the poles in our system is as follows:

$$s_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm \sqrt{1-4KT}}{2T}. \tag{40}$$

If we want to determine, for which range of values of the parameters K and T the poles of the closed-loop system transfer function $G_r(s)$ are in the left half-plane of the complex s-plane, then the condition written below needs to be completed:

$$Re\{s_{1,2}\} < 0 \tag{41}$$

If $\Delta < 0$, then $\sqrt{\Delta}$ will be a part of imaginary element of the poles. Therefore:

$$Re\{s_{1,2}\} < 0 \tag{42}$$

$$\frac{-1}{2T} < 0 \tag{43}$$

$$T > 0 \tag{44}$$

From this case we obtain that $T > 0$. If $\Delta = 0$, the same condition ($T > 0$) is similarly obtained. For $\Delta > 0$:

$$\frac{-1 \pm \sqrt{1-4KT}}{2T} < 0 \tag{45}$$

$$(-1 \pm \sqrt{1-4KT})4T^2 < 0 \tag{46}$$

$4T^2$ is always positive, so it doesn't change the result of the inequality.

$$-1 \pm \sqrt{1-4KT} < 0 \tag{47}$$

$$\pm \sqrt{1-4KT} < 1 \tag{48}$$

As $-\sqrt{1-4KT} < \sqrt{1-4KT}$, we will consider only $\sqrt{1-4KT}$ in the inequality (if it is satisfied for the higher value, then for a smaller value it will be also satisfied):

$$\sqrt{1-4KT} < 1 \tag{49}$$

$$1-4KT < 1 \tag{50}$$

$$4KT > 0 \tag{51}$$

$$KT > 0 \tag{52}$$

From previous case we know that T has to be greater than 0, so to satisfy the inequality K also has to be greater than zero.
The conclusion: the poles of the closed-loop system are in the left half-plane of the complex s-plane for $T > 0$ and $K > 0$.

2.5 Further comparisons

All pairs of $(K_i; T_i)$ - $T > 0$ - for which the step response function of the closed-loop system has the same maximum overshoot σ_m as for the values of the parameters given in task 1.2 need to be found. σ_m depends only on ξ, therefore in our desired system $\xi = \frac{\sqrt{3}}{6}$ as well. In our system:

$$G_r(s) = \frac{K}{Ts^2 + s + K} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}} \tag{53}$$

we know that:

$$\omega_n = \sqrt{\frac{K}{T}} \tag{54}$$

$$2\xi\omega_n = \frac{1}{T} \tag{55}$$

Therefore:

$$2\frac{\sqrt{3}}{6}\sqrt{\frac{K}{T}} = \frac{1}{T} \tag{56}$$

$$\frac{\sqrt{3}}{3}\sqrt{\frac{K}{T}} = \frac{1}{T} \tag{57}$$

$$\frac{1}{3}\frac{K}{T} = \frac{1}{T^2} \tag{58}$$

$$\frac{1}{3}K = \frac{1}{T} \tag{59}$$

$$K = \frac{3}{T} \tag{60}$$

The conditions $T > 0$ and $K > 0$ need to be satisfied. We choose $T_2 = 3T_1 = 3$ and determine $K_2 = \frac{3}{T_2} = \frac{3}{3} = 1$. The corresponding closed-loop system transfer function $G_{r,2}(s)$ has the form:

$$G_{r,2}(s) = \frac{1}{3s^2 + s + 3} \tag{61}$$

The step-response function of $G_{r,1}(s)$ and $G_{r,2}(s)$ have been plotted on the same figure.

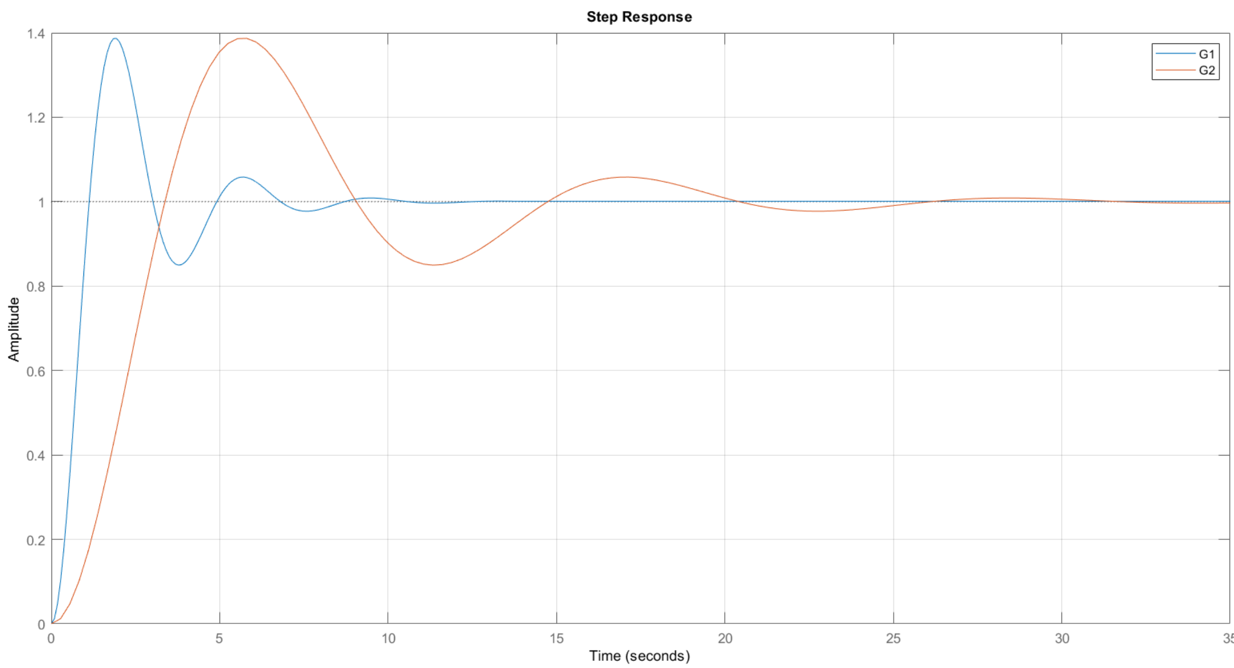

Figure 8: Step responses for both systems

Both systems have - as assumed - the same maximum overshoot. $G_{r,2}(s)$ has a greater time to maximum overshoot t_m, rise time t_r and settling time $t_{1\%}$.

2.6 On the edge of oscillatory and aperiodic behaviour

We assume that T = T_3 =1.

$$G_r(s) = \frac{K}{s^2 + s + K} \tag{62}$$

The poles of this transfer function are:

$$s_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1 \pm \sqrt{1-4K}}{2}. \tag{63}$$

The closed-loop step response function is exactly on the limit between the aperiodic (without overshoot) and oscillatory (with overshoot) behavior when $\Delta = 0$. Therefore:

$$1 - 4K = 0 \tag{64}$$

$$K = \frac{1}{4} = K_3 \tag{65}$$

On the same figure the step response functions of $G_{r,1}(s)$ and $G_{r,3}(s)$ have been plotted:

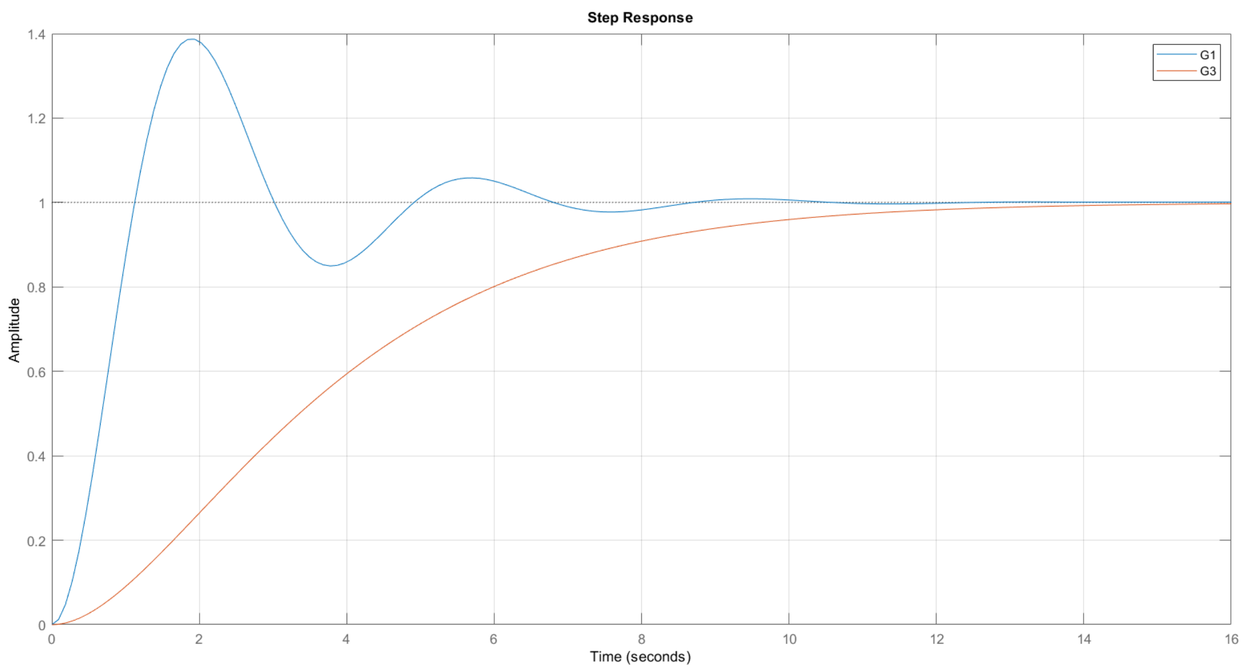

Figure 9: Step responses for functions of $G_{r,1}(s)$ and $G_{r,3}(s)$