STABILITY OF LINEAR SYSTEMS
5^{th} homework

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1 Assignment [1]

A system is described by the following block diagram:

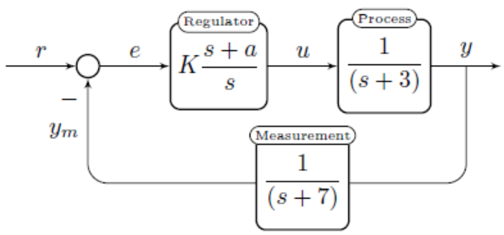

Figure 1: Closed-loop system

1.1 Hurwitz criterion

To use the Hurwitz criterion, transfer function needs to be determined first:

$$G(s) = \frac{K\frac{s+a}{s}\frac{1}{s+3}}{K\frac{s+a}{s}\frac{1}{s+3}\frac{1}{s+7}+1} = \frac{K(s+a)(s+7)}{K(s+a)+s(s+3)(s+7)} = \frac{K(s+a)(s+7)}{s^3+10s^2+(21+K)s+Ka}. \tag{1}$$

Then we create a Hurwitz matrix:

$$\begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 21+K & Ka & 0 \\ 1 & 10 & 21+K \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The Hurwitz criterion is applied:

$$\Delta_1 > 0 \tag{4}$$

$$\Delta_2 > 0 \tag{5}$$

$$\Delta_3 > 0 \tag{6}$$

The following inequalities are obtained:

$$21 + K > 0 \tag{7}$$

$$10(21 + K) - Ka > 0 \tag{8}$$

$$10(21 + K) \cdot 1 + 0 + 0 - 0 - Ka - 0 > 0 \tag{9}$$

The region in the K-a plane for which the closed-loop system is stable is decribed by the inequalities:

$$K > -21 \tag{10}$$

$$210 > K(a - 10) \tag{11}$$

Additionally, all of the coefficients in characteristic equation need to be positive, so we need to add one more condition:

$$Ka > 0 \tag{12}$$

The solution presented graphically can be found below.

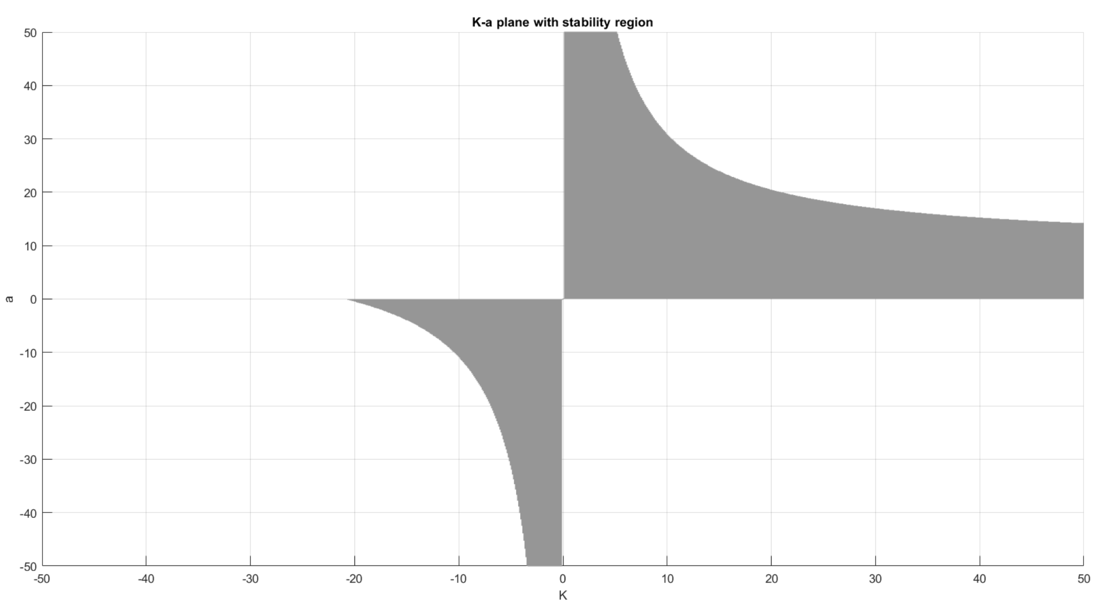

Figure 2: Stability region

1.2 System behaviour

Step response functions for each set of parameters have been plotted on the same figure. For K = 30, a = 20 the system is unstable, for K = 10, a = 31 the system is borderline stable, while for K = 10, a = 10 we obtain a stable system.

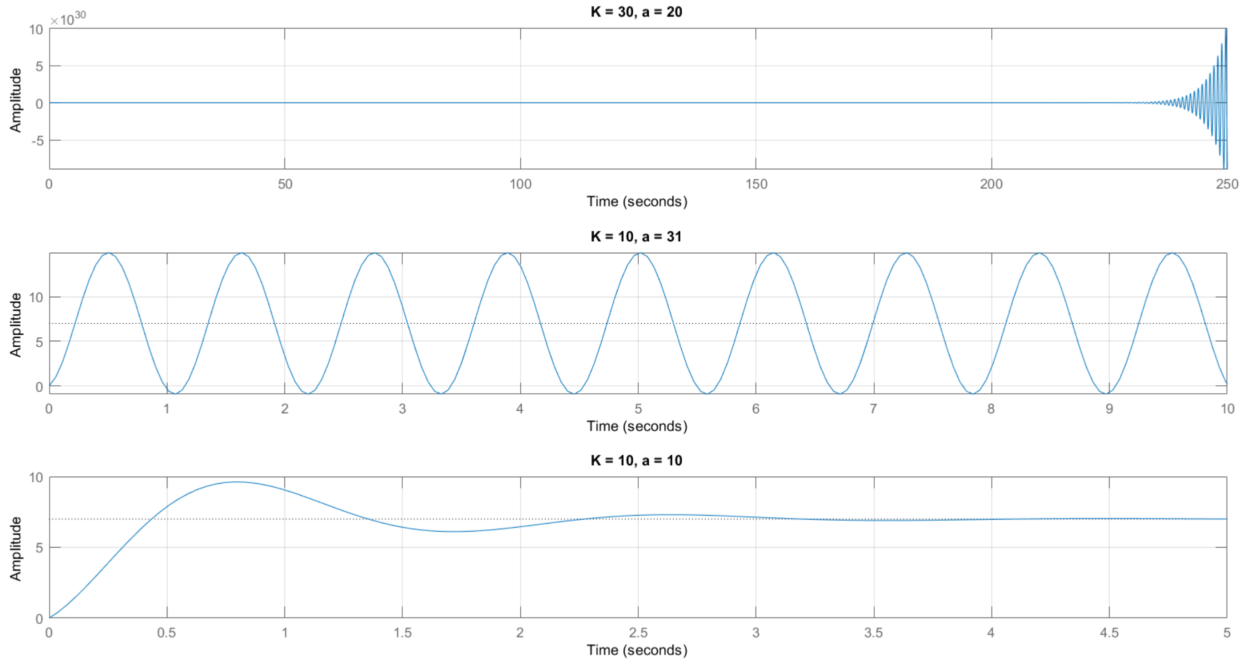

Figure 3: System behaviour

1.3 Regulation error

At the beginning the Laplace transform needs to be determined:

$$E(s) = \frac{1}{1 + K\frac{s+a}{s(s+3)(s+7)}} R(s) = \frac{s(s+3)(s+7)}{s(s+3)(s+7) + K(s+a)} R(s) \tag{13}$$

The regulation error in steady state e_∞ for the input R(s) = $\frac{5}{s}$ with K = 10 and a = 10 needs to be calculated:

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s(s+3)(s+7)}{s(s+3)(s+7) + K(s+a)} R(s)s = \lim_{s\to 0} \frac{s(s+3)(s+7)}{s(s+3)(s+7) + 10(s+10)} 5 = 0 \tag{14}$$

1.4 Regulation error II

Using the above E(s), the regulation error in steady state e_∞ for the input R(s) = $\frac{10}{s^2}$ with K = 10 and a = 10 needs to be calculated:

$$e_\infty = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s(s+3)(s+7)}{s(s+3)(s+7)+K(s+a)} R(s) s = \lim_{s \to 0} \frac{s(s+3)(s+7)}{s(s+3)(s+7)+10(s+10)} \frac{10}{s} = \frac{3 \cdot 7}{10 \cdot 10} 10 = 2.1 \tag{15}$$

1.5 Simulink model

A Simulink model of the system has been created.

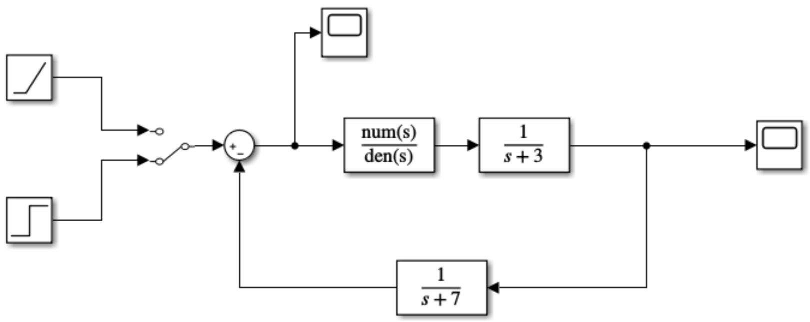

Figure 4: Simulink system

Simulation for input signals and parameters from tasks 1.4 and 1.5 are presented below. It can be seen that for a step input the system's error stabilizes at 0, while for a ramp input it approaches 2.1 (as calculated in previous tasks).

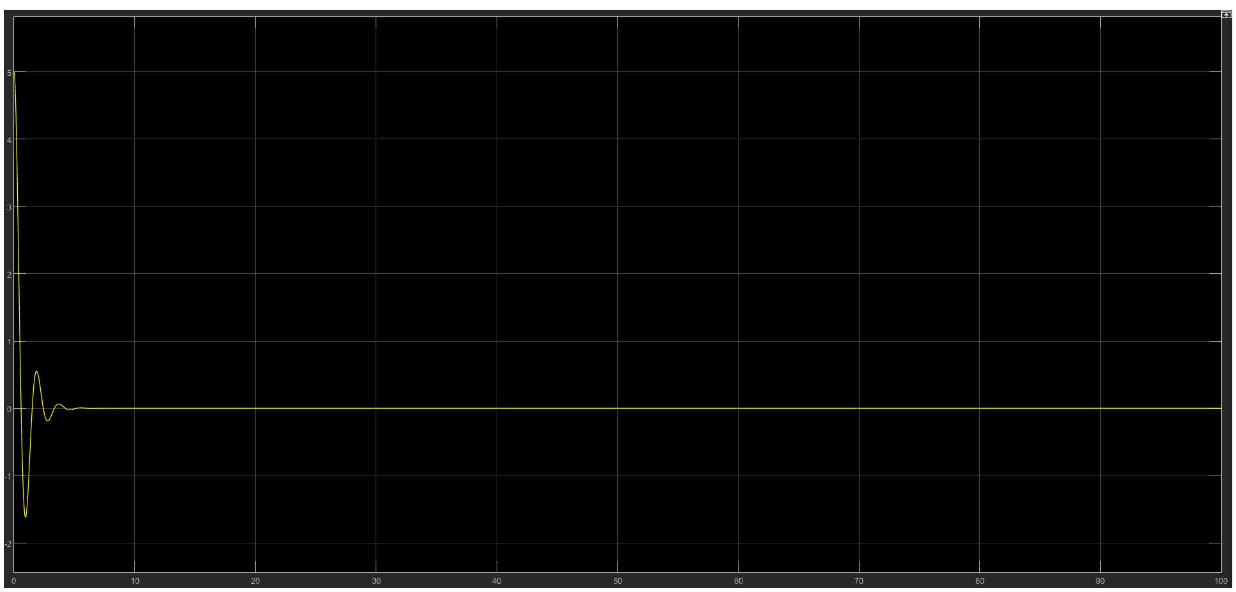

Figure 5: Response for a step input function

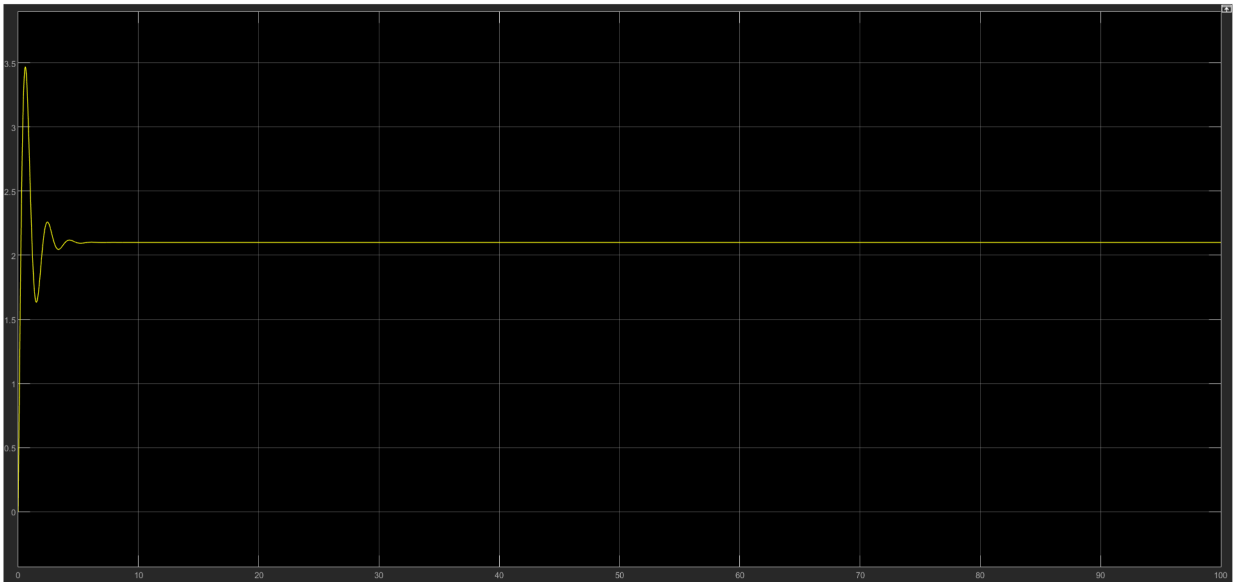

Figure 6: Response for a ramp input function

2 Assignment [2]

A system is described by the following block diagram:

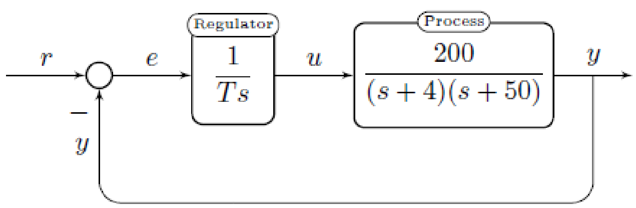

Figure 7: Closed-loop system

2.1 Determining the time constant

Closed-loop system's overshoot of step response function is set to $\sigma_m = 4\%$. To determine a time constant T the following dependence will be used:

$$\gamma \approx 70^o - \sigma_m[\%] = 70^o - 4 = 66^o. \tag{16}$$

Phase margin which has been calculated occurs for a situation where the amplitude characteristics function crosses the frequency axis. Therefore the following equations can be created:

$$|G(j\omega_c)| = 1 \tag{17}$$

$$arg(G(j\omega_c)) = -114^o \tag{18}$$

The next step will be to solve them:

$$G(j\omega) = \frac{200}{Tj\omega}\frac{1}{(j\omega+4)(j\omega+50)} = \frac{-200j}{T\omega}\frac{1}{200-\omega^2+54j\omega} = \frac{-200j}{T\omega}\frac{200-\omega^2-54j\omega}{\omega^4+2516\omega^2+40000} \tag{19}$$

$$arctg\left(\frac{200({\omega_c}^2-200)}{200(-54\omega_c)}\right) = -114^o \iff \omega_c = -122.913 \vee \omega_c = 1.62717 \implies \omega_c = 1.62717 \tag{20}$$

$$\sqrt{\frac{(200(-54\omega_c))^2+(200({\omega_c}^2-200))^2}{(T\omega_c({\omega_c}^4+2516{\omega_c}^2+40000))^2}} = 1 \iff \frac{\sqrt{(200(-54\omega_c))^2+(200({\omega_c}^2-200))^2}}{T\omega_c({\omega_c}^4+2516{\omega_c}^2+40000)} = 1 \iff \tag{21}$$

$$\sqrt{(200(-54\cdot 1.62717))^2+(200((1.62717)^2-200))^2} = 1.62717T((1.62717)^4+2516\cdot(1.62717)^2+40000)$$

$$\iff T = 0.5689$$

Therefore, the time constant T equals 0.5689.

2.2 Gain and phase margins

With T=0.5689 the values of gain and phase margins are computed:

- phase margin

$$\omega_c = 1.62717 \tag{22}$$

$$\gamma = -180^o + arctg\left(\frac{200((1.62717)^2-200)}{200(-54\cdot 1.62717)}\right) = -180^o + 66^o = 114^o \tag{23}$$

- gain margin

$$arctg\left(\frac{200({\omega_\pi}^2-200)}{200(-54\omega_\pi)}\right) = -180^o \iff arctg\left(\frac{200({\omega_\pi}^2-200)}{200(-54\omega_\pi)}\right) = 0^o \iff \frac{200({\omega_\pi}^2-200)}{200(-54\omega_\pi)} = 0 \implies \omega_\pi = \sqrt{200} \tag{24}$$

$$A_r = \frac{\sqrt{(200(-54\sqrt{200}))^2+(200(\sqrt{200}^2-200))^2}}{0.5689\sqrt{200}(\sqrt{200}^4+2516\sqrt{200}^2+40000)} \approx 30.72 \tag{25}$$

2.3 Nyquist diagram

The transfer function for T = 0.5689 of the open-loop system is as below:

$$G(j\omega) = \frac{200}{0.5689\omega}\frac{-54\omega+j(\omega^2-200)}{\omega^4+2516\omega^2+40000} \tag{26}$$

Firstly, intersections with axes need to be calculated:

$$Re\{G(j\omega)\} = 0 \iff \omega \in \emptyset \tag{27}$$

$$Im\{G(j\omega)\} = 0 \Longleftrightarrow \omega = \pm\sqrt{200} \tag{28}$$

$$G(j\sqrt{200}) = \frac{-1}{54T} = \frac{-1}{54T} \approx -0.03255 \tag{29}$$

It is also necessary to calculate limes for 0+ (and later see for 0^-) as the transfer function has a pole in the origin.

$$\lim_{\omega\to 0^+} Re\{G(j\omega)\} = -0.47 \tag{30}$$

$$\lim_{\omega\to 0^+} Im\{G(j\omega)\} = -\infty \tag{31}$$

Full Nyquist diagram (positive and negative frequencies) of the open-loop system is sketched as below.

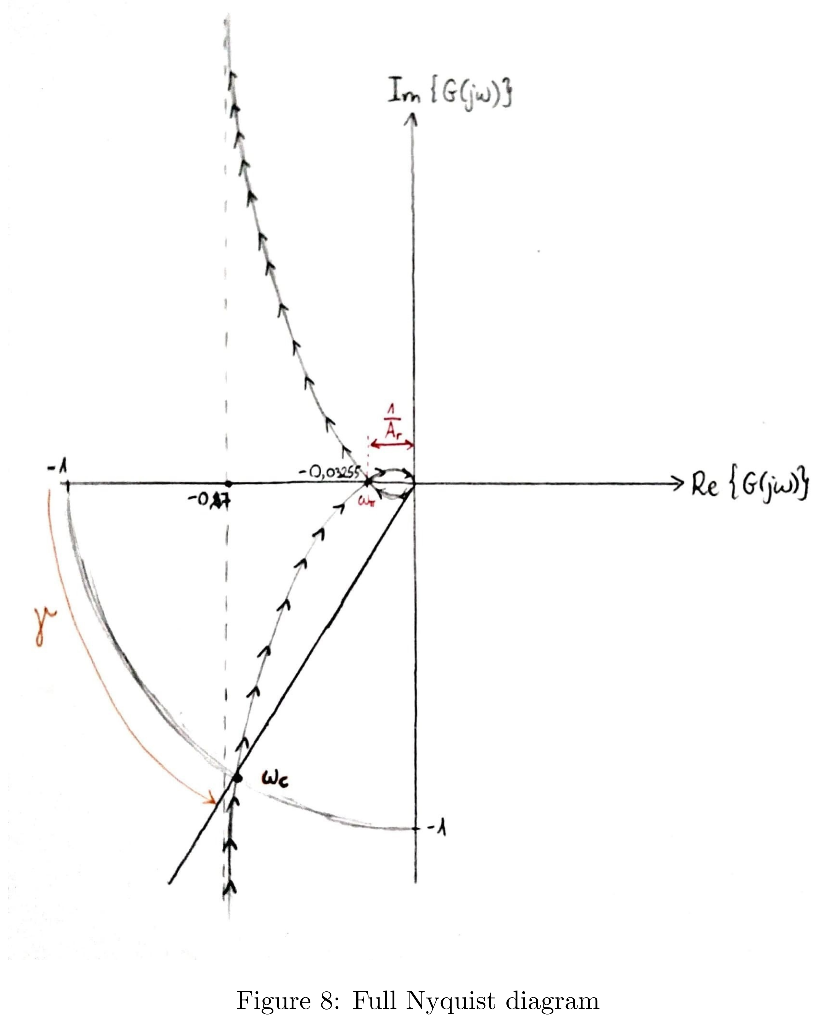

Figure 8: Full Nyquist diagram

In the open-loop system there are no poles in the right half-plane, so the closed-loop system is stable when $Re\{G(j\omega)\} = 0$ and $Im\{G(j\omega)\} < -1$. It is already calculated that then $Im\{G(j\omega)\} = \frac{-1}{54T}$, so for $T > 0$:

$$\frac{-1}{54T} < -1, \tag{32}$$

$$\frac{1}{54T} > 1, \tag{33}$$

$$\frac{1}{54} > T. \tag{34}$$

The above stated condition has to be met for the closed-loop system to be stable.

2.4 Step response

Using the approximative relationships between frequency and time domain and the formula $\omega_c = \frac{3}{t_m}$ the time of the first maximum of the step-response function can be determined:

$$t_m = \frac{3}{\omega_c} = \frac{3}{1.62717} \approx 1.8437[s] \tag{35}$$

Using Matlab the step response of the closed-loop system has been simulated and compared the time of the first maximum obtained in the simulation to the one obtained from approximative calculations. There is a slight difference, which might be due to the inaccuracy of T calculation or the inaccuracy of used approximation.

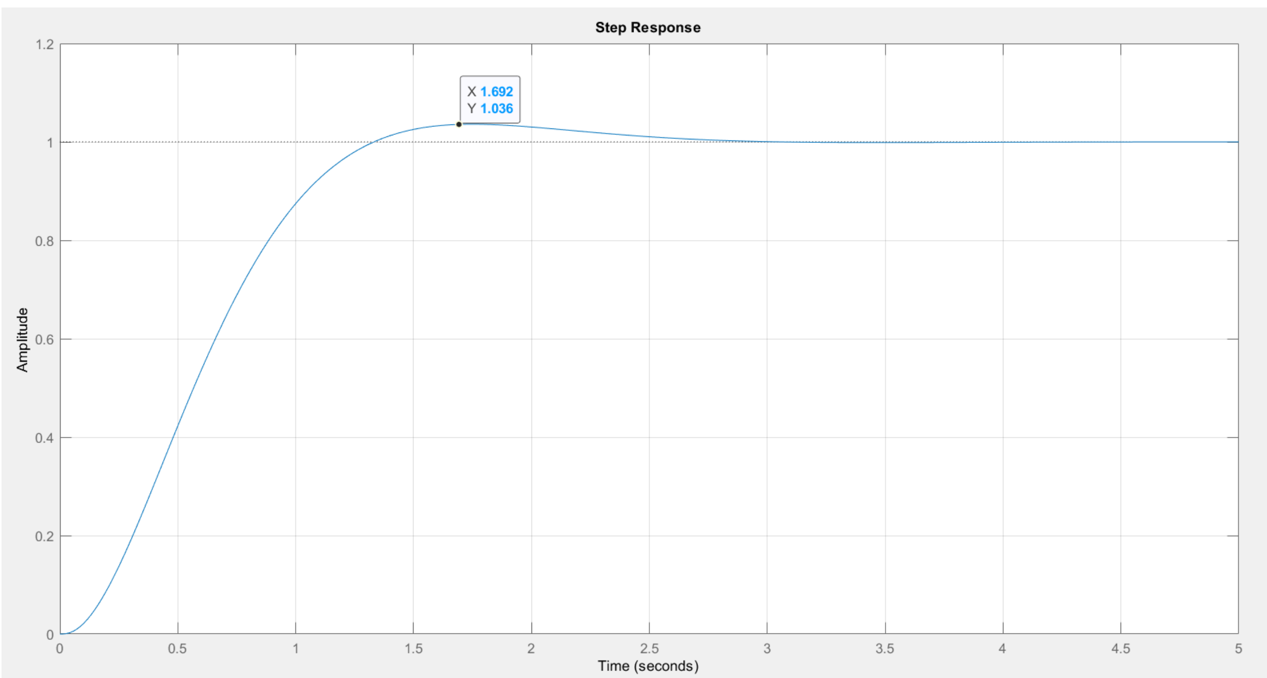

Figure 9: Step response

2.5 Critical open-loop gain

The value of the critical open-loop gain for which the closed-loop system is borderline stable is 34th inequality - it is borderline stable for $T = \frac{1}{54}$. The corresponding frequency is equal to $\omega_\pi = \sqrt{200}$ and period of oscillations at the output of the system is $t_\pi = \frac{1}{\omega_\pi} =\approx 0.0707[s]$.