PID REGULATOR
6^{th} homework

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22.12.2020r.

1 Assignment [1]

A closed-loop control system is described by the following block diagram:

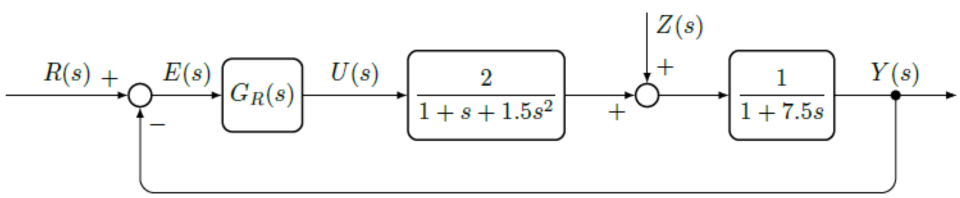

Figure 1: Closed-loop system

Regulators used in the assignment and their corresponding transfer functions are summarized below:

- PI: $G_R(s) = K_R(1 + \frac{1}{T_I s})$,
- PID (ideal): $G_R(s) = K_R(1 + \frac{1}{T_I s} + T_D s)$,
- PID (real): $G_R(s) = K_R(1 + \frac{1}{T_I s} + \frac{T_D s}{1+T_\nu s})$.

1.1 Parametrizing a PI regulator with respect to r(t)

We assume the following substitutions:

- $G_R(s) = K_R(1 + \frac{1}{T_I s})$,
- $G_1(s) = \frac{2}{1.5s^2+s+1}$,
- $G_2(s) = \frac{1}{7.5s+1}$.

The error signal with respect to a step reference signal r(t) needs to be calculated:

$$E(s) = R(s) - Y(s), \tag{1}$$

$$E(s) = R(s) - \frac{G_R(s)G_1(s)G_2(s)}{1 + G_R(s)G_1(s)G_2(s)}R(s), \tag{2}$$

$$E(s) = \frac{1}{1 + G_R(s)G_1(s)G_2(s)}R(s), \tag{3}$$

$$E(s) = \frac{1}{1 + K_R(1 + \frac{1}{T_I s})\frac{2}{1.5s^2+s+1}\frac{1}{7.5s+1}}\frac{1}{s}, \tag{4}$$

$$E(s) = \frac{1}{s + K_R(s + \frac{1}{T_I})\frac{2}{1.5s^2+s+1}\frac{1}{7.5s+1}}, \tag{5}$$

$$E(s) = \frac{1}{s + \frac{2K_R(s+\frac{1}{T_I})}{11.25s^3+9s^2+8.5s+1}}, \tag{6}$$

$$E(s) = \frac{11.25s^3 + 9s^2 + 8.5s + 1}{11.25s^4 + 9s^3 + 8.5s^2 + (1 + 2K_R)s + 2\frac{K_R}{T_I}}, \tag{7}$$

$$E(s) = \frac{11.25T_I s^3 + 9T_I s^2 + 8.5T_I s + T_I}{11.25T_I s^4 + 9T_I s^3 + 8.5T_I s^2 + (1 + 2K_R)T_I s + 2K_R}, \tag{8}$$

Obtained E(s) has a form $E(s) = \frac{c_0+c_1s+...+c_{n-1}s^{n-1}}{d_0+d_1s+...+d_ns^n}$ and is of the 4th order. Therefore, the ISE criterion has the form:

$$I_{3,4} = \frac{c_3^2(-d_0^2d_3 + d_0d_1d_2) + (c_2^2 - 2c_1c_3)d_0d_1d_4 + (c_1^2 - 2c_0c_2)d_0d_3d_4 + c_0^2(-d_1d_4^2 + d_2d_3d_4)}{2d_0d_4(-d_0d_3^2 - d_1^2d_4 + d_1d_2d_3)} \tag{9}$$

Partial derivatives need to be calculated and equated to zero. The solutions can be found with the script presented at the end of this assignment. There are two solutions: for $K_R \approx 14.926$ and $T_I \approx 18.006$ the model doesn't act as a PI regulator, so the values $K_R \approx 1.417$ and $T_I \approx 8.484$ are the ones we were looking for.

1.2 Simulation of behaviour

The behavior of the closed-loop control system from the subtask [1.1] with the reference signal $r(t) = S(t)$ and the disturbance signal $z(t) = S(t - 60)$ was simulated. The obtained control signal u(t) and the output signal y(t) were plotted on the same figure.

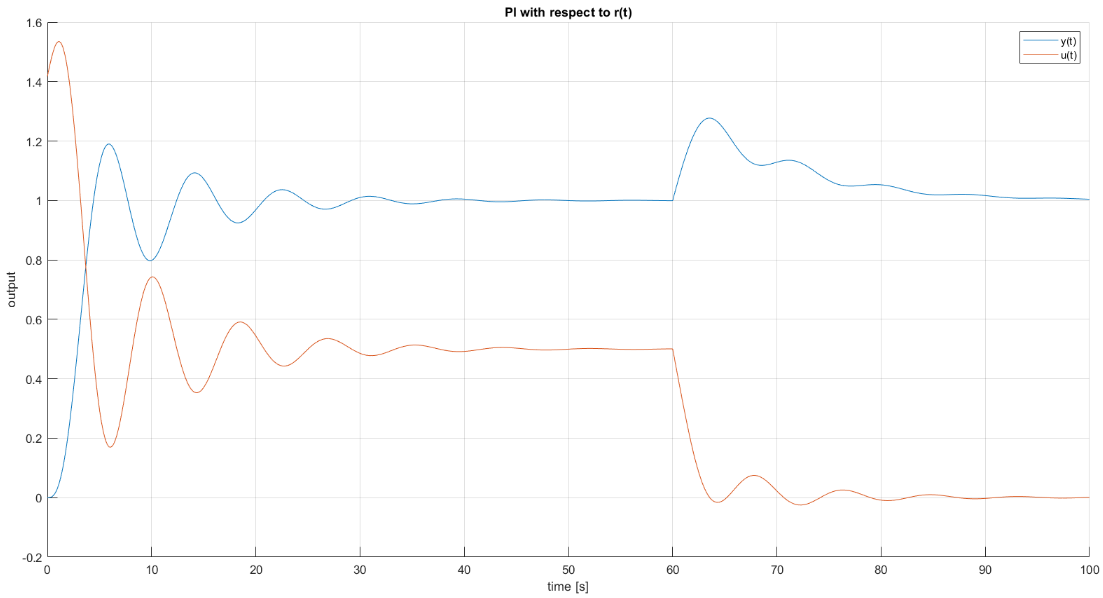

Figure 2: Comparison of u(t) and y(t) of the system.

1.3 Parametrizing a PI regulator with respect to z(t)

We assume the following substitutions:

- $G_R(s) = K_R(1 + \frac{1}{T_I s})$,
- $G_1(s) = \frac{2}{1.5s^2+s+1}$,
- $G_2(s) = \frac{1}{7.5s+1}$.

The error signal with respect to a step reference signal z(t) needs to be calculated:

$$E(s) = -Y(s), \tag{10}$$

$$E(s) = -\frac{G_2(s)}{1 + G_R(s)G_1(s)G_2(s)} Z(s), \tag{11}$$

$$E(s) = -\frac{\frac{1}{7.5s+1}}{1 + K_R(1 + \frac{1}{T_I s})\frac{2}{1.5s^2+s+1}\frac{1}{7.5s+1}} \frac{1}{s}, \tag{12}$$

$$E(s) = -\frac{1.5s^2 + s + 1}{(1.5s^2 + s + 1)(7.5s + 1) + 2K_R(1 + \frac{1}{T_I s})} \frac{1}{s}, \tag{13}$$

$$E(s) = -\frac{1.5T_I s^2 + T_I s + T_I}{11.25T_I s^4 + 9T_I s^3 + 8.5T_I s^2 + (1 + 2K_R)T_I s + 2K_R}, \tag{14}$$

Obtained E(s) has a form $E(s) = \frac{c_0+c_1s+...+c_{n-1}s^{n-1}}{d_0+d_1s+...+d_ns^n}$ and is of the 4th order. Therefore, the ISE criterion has the form:

$$I_{3,4} = \frac{c_3^2(-d_0^2d_3 + d_0d_1d_2) + (c_2^2 - 2c_1c_3)d_0d_1d_4 + (c_1^2 - 2c_0c_2)d_0d_3d_4 + c_0^2(-d_1d_4^2 + d_2d_3d_4)}{2d_0d_4(-d_0d_3^2 - d_1^2d_4 + d_1d_2d_3)} \tag{15}$$

Partial derivatives need to be calculated and equated to zero. The solutions can be found with the script presented at the end of this assignment. There are three solutions: for $K_R \approx 2.379$, $T_I \approx 1.137$ and $K_R \approx 10.191$, $T_I \approx 0.937$ the model doesn't act as a PI regulator (is unstable), so the values $K_R \approx 1.732$ and $T_I \approx 4.186$ are the ones we were looking for.

1.4 Simulation of behaviour II

The behavior of the closed-loop control system from the subtask [1.3] with the reference signal $r(t) = S(t)$ and the disturbance signal $z(t) = S(t - 60)$ was simulated. The obtained control signal u(t) and the output signal y(t) were plotted on the same figure.

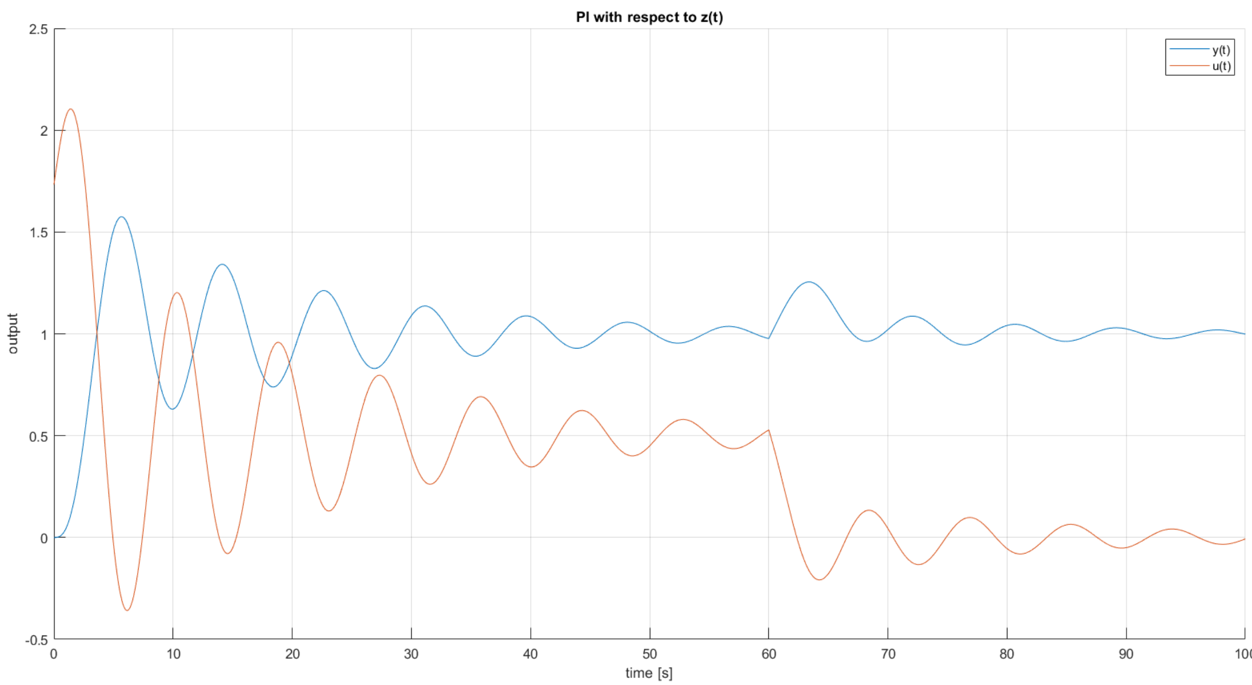

Figure 3: Comparison of u(t) and y(t) of the system.

1.5 Comparison of both behaviours

The results from subtasks [1.2] and [1.4] are presented below so that one figure shows the outputs signal from the process y(t) and the other figure shown the control signals u(t).

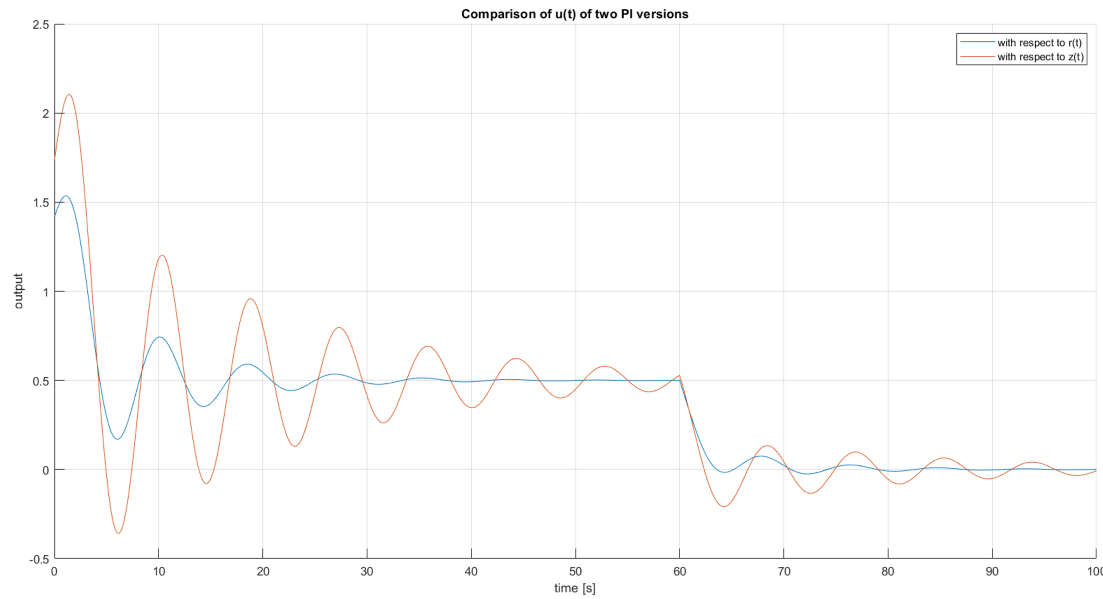

Figure 4: Comparison of inputs.

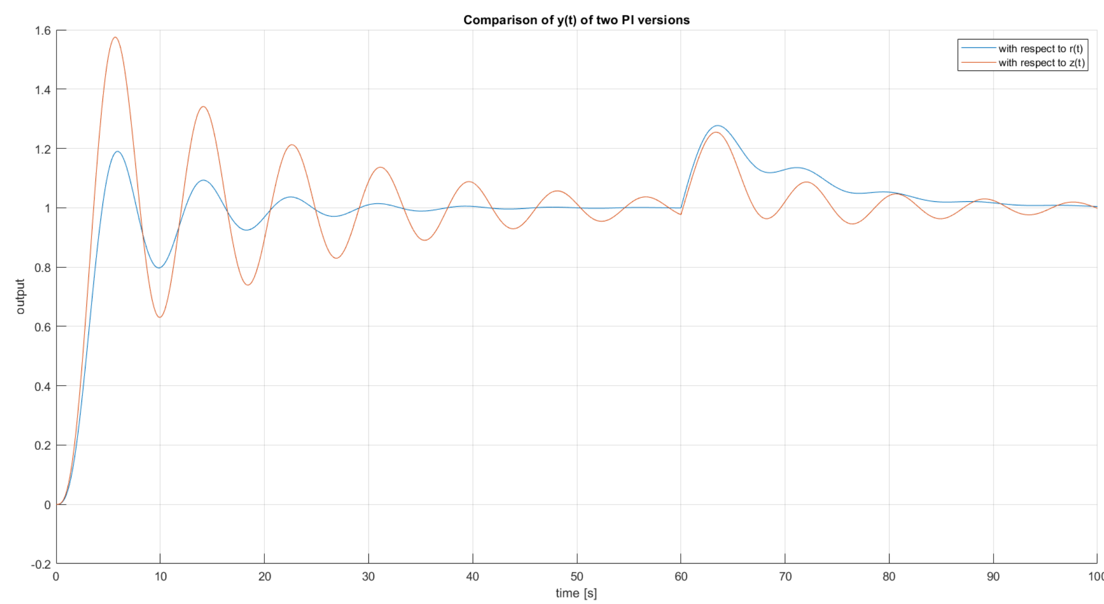

Figure 5: Comparison of outputs.

The regulator parametrized with respect to reference signal r(t) has a smaller settling time with respect to the reference signal and The regulator parametrized with respect to disturbance signal z(t) has a better compensation of the disturbance.

1.6 Ziegler-Nichols method

A PI and an ideal PID regulator need to be parametrized by using the Ziegler-Nichols method based on the stability margin. Therefore, G_R is substituted with P element (K_R) and the open-loop system has to be analyzed using Nyquist criterion:

$$G(s) = \frac{2K_R}{11.25s^3 + 9s^2 + 8.5s + 1} \tag{16}$$

$$G(j\omega) = \frac{2K_R}{-11.25j\omega^3 - 9\omega^2 + 8.5j\omega + 1} \tag{17}$$

$$G(j\omega) = \frac{2K_R}{-11.25j\omega^3 - 9\omega^2 + 8.5j\omega + 1} \tag{18}$$

$$G(j\omega) = \frac{2K_R}{1 - 9\omega^2 + j\omega(8.5 - 11.25\omega^2)} \tag{19}$$

$$G(j\omega) = \frac{2K_R(1 - 9\omega^2 - j\omega(8.5 - 11.25\omega^2))}{(1 - 9\omega^2)^2 + (\omega(8.5 - 11.25\omega^2))^2} \tag{20}$$

The value of gain for which permanent oscillations are produced in the closed-loop system needs to be determined, therefore from Nyquist criterion:

$$Im\{G(j\omega)\} = 0 \Longleftrightarrow \frac{-2K_R\omega(8.5 - 11.25\omega^2)}{(1 - 9\omega^2)^2 + (\omega(8.5 - 11.25\omega^2))^2} = 0 \Longleftrightarrow \omega = 0 \text{ or } \omega = \sqrt{\frac{34}{45}} \tag{21}$$

For $\omega = 0$, Re{ G(jω)} = $2K_R$. It would have to be equal to -1 if we wanted to have the system on the edge of stability, but K_R has to be positive, so we cannot take this value. For $\omega = \sqrt{\frac{34}{45}}$, Re{ G(j$\omega$)} = $\frac{-10K_R}{29}$. It would have to be equal to -1 if we wanted to have the system on the edge of stability, so it can be calculated that $K_R = 2.9$. At the end we obtain $K_{Rkr} = 2.9$ and $T_{kr} = \frac{2\pi}{\omega} \approx 7.2285$. Therefore, using Ziegler-Nichols rule, the following parameters are obtained:

- PI regulator: $K_R = 0.45K_{Rkr} = 1.305$, $T_I = 0.85T_{kr} \approx 6,1442$,
- ideal PID regulator: $K_R = 0.6K_{Rkr} = 1.74$, $T_I = 0.5T_{kr} \approx 3.6143$, $T_D = 0.12T_{kr} = 0.8674$.

1.7 Real PID regulator

The ideal PID regulator parameterized in task [1.6] is expanded with a small time constant $T_\nu = 0.15T_D \approx 0.1301$ so that a real PID regulator is obtained:

$$G_R(s) = 1.74\left(1 + \frac{1}{3.6143s} + \frac{0.8674s}{1 + 0.1301s}\right). \tag{22}$$

1.8 Simulation of behaviour III

The behavior of the closed-loop control system with a PI regulator from subtask [1.6] and with a real PID regulator from task [1.7] with the reference signal $r(t) = S(t)$ and the disturbance signal $z(t) = S(t - 60)$ was simulated. The output signals y(t) were plotted on one figure and the control signals u(t) on the other figure.
It can be seen that PID regulator better deals with compensating the disturbance and has a smaller settling time but on the other hand has a bigger overshoot and demands more energy put into the system than PI regulator.

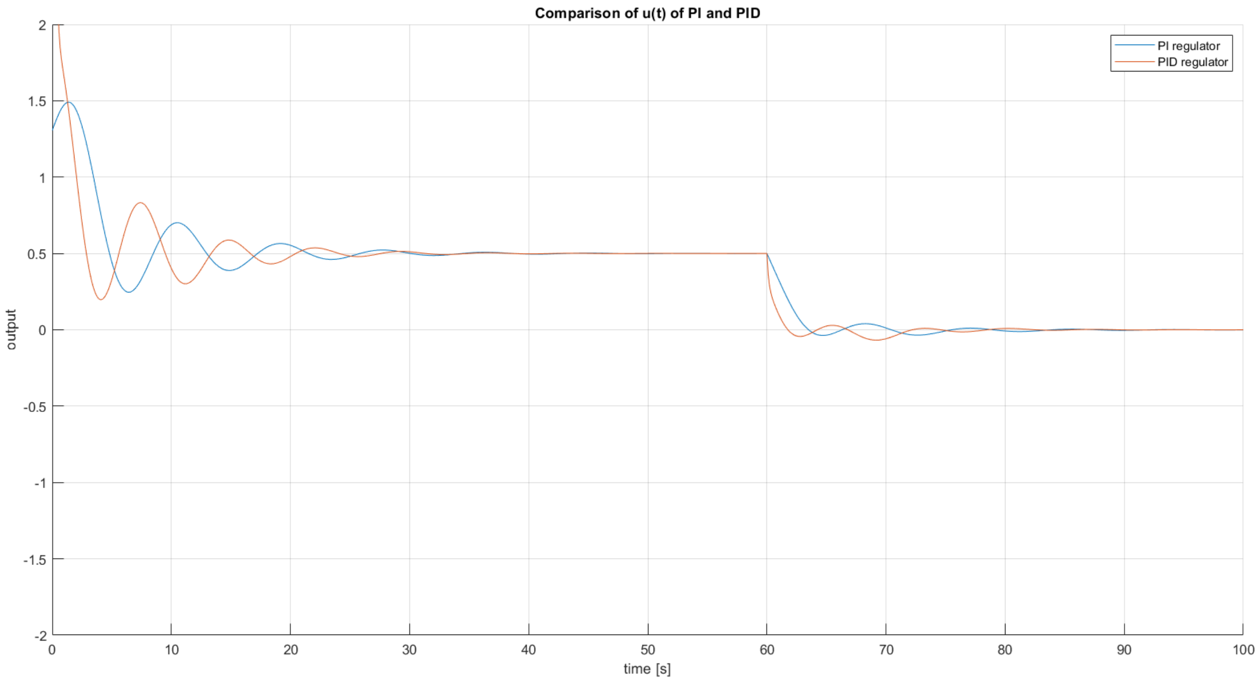

Figure 6: Comparison of u(t).

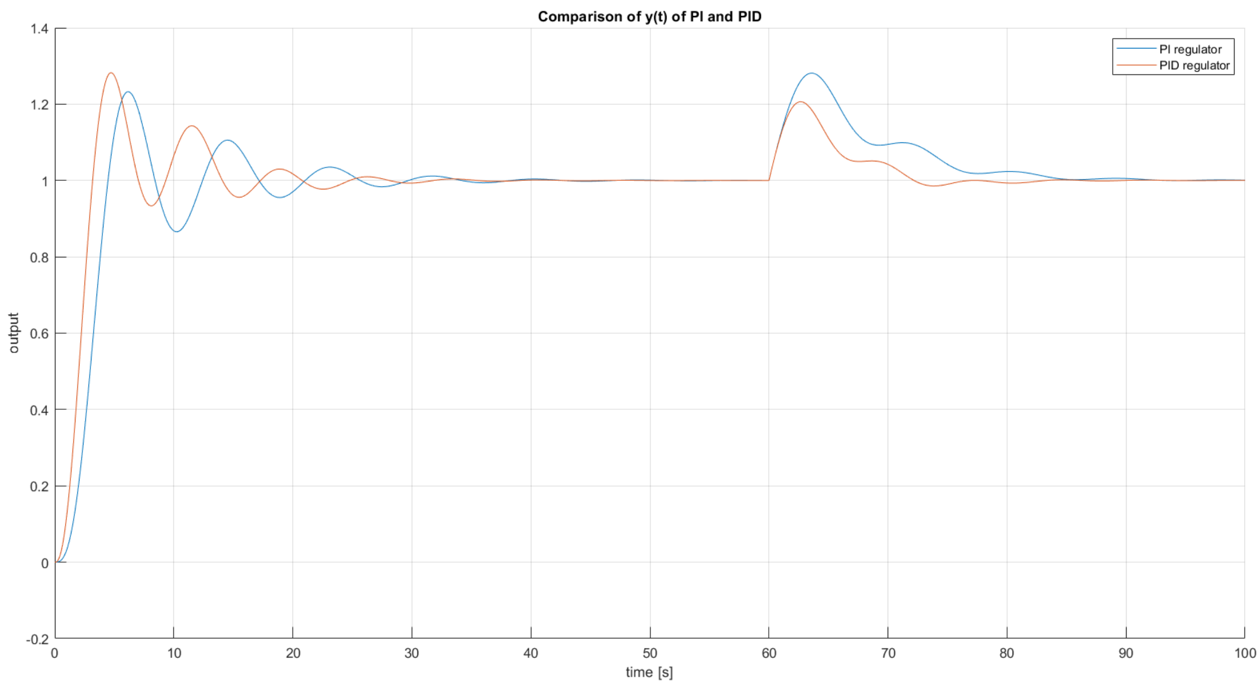

Figure 7: Comparison of y(t).

1.9 Matlab script & Simulink model

```
close all;
clear all;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

sim_time = 100;

r_steptime = 0;
r_delta = 1;
r_initvalue = 0;

z_steptime = 60;
z_delta = 1;
z_initvalue = 0;

max_step_size = 0.1;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PI element v1 - derivatives

syms T_I K_R

d_0 = 2*K_R;
```

```
d_1 = (1 + 2*K_R)*T_I;
d_2 = 8.5*T_I;
d_3 = 9*T_I;
d_4 = 11.25*T_I;

c_0 = T_I;
c_1 = 8.5*T_I;
c_2 = 9*T_I;
c_3 = 11.25*T_I;

I_4 = (c_3^2*(-d_0^2*d_3 +d_0*d_1*d_2) +(c_2^2-2*c_1*c_3)*d_0*d_1*d_4 + (c_1^2-2*c_0*c_2)*d_0*d_3*d_4 +
     c_0^2*(-d_1*d_4^2 + d_2*d_3*d_4))...
    /(2*d_0*d_4*(-d_0*d_3^2 - d_1^2*d_4 + d_1*d_2*d_3));

dT_I = diff(I_4, T_I);
dK_R = diff(I_4, K_R);

S = vpasolve([dT_I == 0, dK_R == 0], [T_I, K_R], [0+1e-10, inf; 0+1e-10, inf;]);

K_R1 = double(S.K_R(1));
K_R2 = double(S.K_R(2));

T_I1 = double(S.T_I(1));
T_I2 = double(S.T_I(2));


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PI element v1 - figures
K_r = K_R1;
T_I = T_I1;
T_D = 0;
T_V = 0;

sim('model1');
figure(1); hold on;
plot(ans.t, ans.y);
plot(ans.t, ans.u);
grid on;
xlabel('time [s]');
ylabel('output');
legend('y(t)', 'u(t)');
title('PI with respect to r(t)');

figure(3); hold on;
plot(ans.t, ans.y);

figure(4); hold on;
plot(ans.t, ans.u);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%PI element v2 - derivatives

syms T_I K_R

d_0 = 2*K_R;
d_1 = (1 + 2*K_R)*T_I;
d_2 = 8.5*T_I;
```

```
d_3 = 9*T_I;
d_4 = 11.25*T_I;

c_0 = -T_I;
c_1 = -T_I;
c_2 = -1.5*T_I;
c_3 = sym('0');

I_4 = (c_3^2*(-d_0^2*d_3 +d_0*d_1*d_2) +(c_2^2-2*c_1*c_3)*d_0*d_1*d_4 + (c_1^2-2*c_0*c_2)*d_0*d_3*d_4 +
     c_0^2*(-d_1*d_4^2 + d_2*d_3*d_4))...
    /(2*d_0*d_4*(-d_0*d_3^2 - d_1^2*d_4 + d_1*d_2*d_3));

dT_I = diff(I_4, T_I);
dK_R = diff(I_4, K_R);

S = vpasolve([dT_I == 0, dK_R == 0], [T_I, K_R], [0+1e-10, inf; 0+1e-10, inf;]);

K_R1 = double(S.K_R(1));
K_R2 = double(S.K_R(2));
K_R3 = double(S.K_R(3));

T_I1 = double(S.T_I(1));
T_I2 = double(S.T_I(2));
T_I3 = double(S.T_I(3));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PI element v2 - figures
K_r = K_R2;
T_I = T_I2;
T_D = 0;
T_V = 0;

sim('model1');
figure(2); hold on;
plot(ans.t, ans.y);
plot(ans.t, ans.u);
grid on;
xlabel('time [s]');
ylabel('output');
legend('y(t)', 'u(t)');
title('PI with respect to z(t)');

figure(3); hold on;
plot(ans.t, ans.y);

figure(4); hold on;
plot(ans.t, ans.u);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% comparison of v1 and v2

figure(3); hold on; grid on;
xlabel('time [s]');
ylabel('output');
legend('with respect to r(t)', 'with respect to z(t)');
title('Comparison of y(t) of two PI versions');
```

```
figure(4); hold on; grid on;
xlabel('time [s]');
ylabel('output');
legend('with respect to r(t)', 'with respect to z(t)');
title('Comparison of u(t) of two PI versions');


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% PI element v3
K_r = 1.305;
T_I = 6.1442;
T_D = 0;
T_V = 0;

sim('model1');
figure(5); hold on;
plot(ans.t, ans.y);

figure(6); hold on;
plot(ans.t, ans.u);

% PID element
K_r = 1.74;
T_I = 3.6143;
T_D = 0.8674;
T_V = 0.1301;

sim('model1');
figure(5); hold on;
plot(ans.t, ans.y);

figure(6); hold on;
plot(ans.t, ans.u);


figure(5); hold on; grid on;
xlabel('time [s]');
ylabel('output');
legend('PI regulator', 'PID regulator');
title('Comparison of y(t) of PI and PID');

figure(6); hold on; grid on;
ylim([-2 2]);
xlabel('time [s]');
ylabel('output');
legend('PI regulator', 'PID regulator');
title('Comparison of u(t) of PI and PID');
```

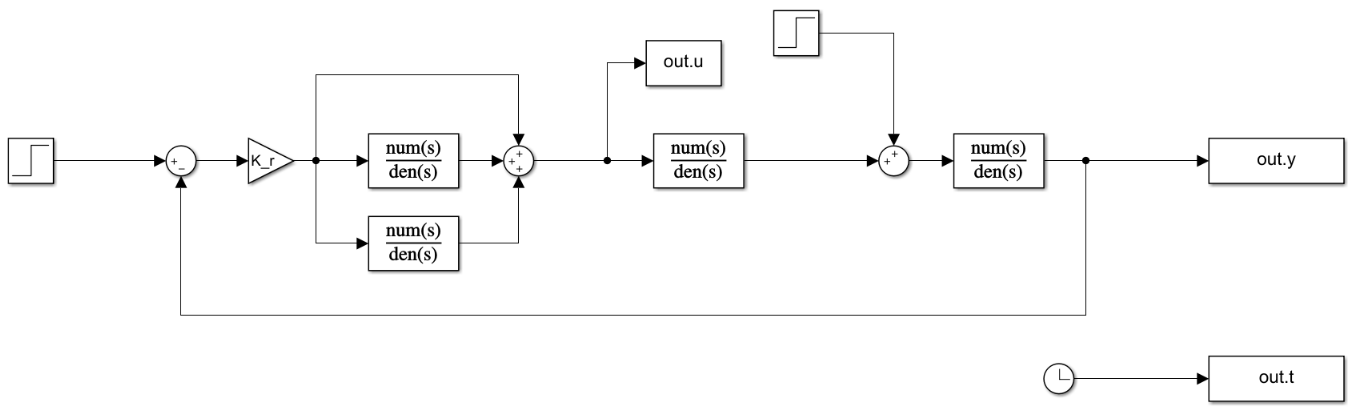

Figure 8: Simulink model.